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Instructions

The exam consists of 5 questions, 1– 5. Each question has three items, a – c.

Within each question:

Item (a) carries a weight of 10 marks.

Item (b) carries a weight of 6 marks.

Item (c) carries a weight of 4 marks.

Answer ALL questions in the spaces provided.

If more space is required, use the back of the PREVIOUS page.

Show all your working and include sketches where appropriate.

Question 1:

Consider the set of 26 lower-case characters, $\mathcal{C} = \{a, b, c, \ldots, z\}$. Let R be a binary relation on $\mathcal{C}^3 := \mathcal{C} \times \mathcal{C} \times \mathcal{C}$ such that $\omega_1 \, R \, \omega_2$ if and only if the word ω_1 appears before the word ω_2 in lexicographic ordering (alphabetical ordering as in a dictionary). Here treat a word as a 3-tuple (c_1, c_2, c_3), not necessarily valid in English or any other human language. That is, any 3-tuple is considered a word.

For example, we have that $\big((x, b, h), (y, b, d)\big) \in R$ and $\big((z, a, c), (y, b, d)\big) \notin R$.
Also $\big((k, b, c), (k, b, c)\big) \notin R$.

(1a) Consider the set,

$$B = \{\omega \in \mathcal{C}^3 \ : \ \big(\omega, (c, a, a)\big) \in R\}.$$

Determine $|B|$.

(1b) For words $\omega_1, \omega_2 \in \mathcal{C}^3$ denote the set,

$$\mathcal{I}_{\omega_1,\omega_2} = \{\omega \in \mathcal{C}^3 \ : \ \omega_1 \, R \, \omega \ \wedge \ \omega \, R \, \omega_2\}.$$

Prove that if $\neg \ \omega_1 \, R \, \omega_2$ then $\mathcal{I}_{\omega_1,\omega_2} = \emptyset$.

(1c) Denote now $\mathcal{D} = \{a, b\}$ (the set of the first two lower-case letters). Determine the value of

$$\big|(\mathcal{D}^3 \times \mathcal{D}^3) \cap R\big|.$$

Question 2:

Let n and k be positive integers with $k \leq n$. Consider the identity (sometimes known as Pascal's formula),

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}.$$

(2a) Prove this identity using only words (without writing any mathematical symbols except for n and k). If you are not able to do that, prove it using mathematical symbols.

(2b) Use induction and the above identity to prove that for $n \geq 0$,

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

(2c) Prove (in any way possible) that for $n \geq 0$,

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0.$$

Question 3:

Given a function, $f : [0, \infty) \to \mathbb{R}$ and a variable $s \in \mathbb{R}$, it is often of interest to evaluate the function, $\widehat{f}(s)$ defined as

$$\widehat{f}(s) = \int_0^\infty e^{sx} f(x)\, dx,$$

when the indefinite integral converges.

(3a) Let $f(x) = 2e^{-2x}$. Determine $\widehat{f}(5)$.

(3b) Continuing with $f(x) = 2e^{-2x}$, for what values of $s \in \mathbb{R}$ does the indefinite integral of $\widehat{f}(s)$ converge?

(3c) For an arbitrary $f(x)$, denote,

$$m_n = \int_0^\infty x^n f(x)\,dx.$$

Assuming you may interchange limits of summations and integrals, use the fact that,

$$e^{sx} = \sum_{k=0}^{\infty} \frac{(sx)^k}{k!},$$

to show that for $n = 0, 1, 2, \ldots$

$$\frac{d\widehat{f}(s)^n}{ds^n}\Big|_{s=0} = m_n.$$

Here the expression means taking the n'th derivative of $\widehat{f}(s)$ and then substituting $s = 0$.

Question 4:

Consider the function,

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-1)^2}{2}}.$$

It is known that,

$$\int_{-\infty}^{\infty} f(x)\,dx = 1.$$

(4a) Does $f(x)$ have a global maximum? If so what is it? Prove/show your result using the first and second derivative.

(4b) Evaluate,

$$\int_{-\infty}^{\infty} x^2 f(x)\, dx.$$

(4c) Determine an interval $[a, b] \subset \mathbb{R}$ such that,

$$[a, b] = \{x \in \mathbb{R} \mid f''(x) \leq 0\}.$$

If no such interval exists, prove it.

Question 5:

Consider a function $F : \mathbb{R}^3 \to \mathbb{R}^{3\times 3}$ defined as follows,

$$F(x_1, x_2, x_3) = \begin{bmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_3^2 \\ 1 & x_3 & x_3^2 \end{bmatrix}.$$

(the third column contains the values squared).

(5a) Determine the matrix $F(y, 0, 0) * F(y, 1, 1)$ (where $*$ is matrix multiplication).

(5b) Is $F(y, 0, 0)$ singular for any value of y? If so for what value? If not, prove your result.

(5c) Consider now the function $f : \mathbb{R} \to \mathbb{R}$ defined by,

$$f(z) = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} * F(0, z, 0) * \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Does $f(z)$ have a global minimum? If so find it.

END OF EXAMINATION