MATH7501 Some Problems for Assignment (4)

Evaluate the following limits:

(a) $\lim_{x\to 0^+}\left(\frac{1}{x}-\frac{1}{\sin x}\right)$

(b) $\lim_{x\to\pi/4}(1-\tan x)(\sec 2x)$

(c) $\lim_{x\to\infty} x(2^{1/x}-1)$ (Hint: Express the limit in terms of $t = 1/x$)

(d) $\lim_{x\to 1} x^{1/(1-x)}$

Let f be a function defined by

$$f(x)=\begin{cases} 2x-x^2, & x\le 1\\ x^2+kx+p, & x>1\end{cases}$$

(e) For what values of k and p will f be continuous and differentiable at $x = 1$?

(f) For the values of k and p found in part (a), on what interval or intervals is f increasing?

(g) Consider the function

$$f(x) = x^3 + \frac{\sin(x)}{x} - 3.$$

Show that $f(x)$ is continuous on $\mathbb{R}$. Does there exist $\alpha \in \mathbb{R}$, such that $f(\alpha) = 17$? If so, state why. Use the bisection method to find $\alpha \in \mathbb{R}$ such that $f(\alpha) = 0$.

(h) Consider the function f acting from $\mathbb{R}^2 \setminus \{0,0\}$ to $\mathbb{R}$ and given by the formula

$$f(x,y) = \frac{1-e^{3x^2-y}}{2x^2+y^4}.$$

Does the limit $\lim_{(x,y)\to(0,0)} f(x,y)$ exist? Justify your answer.

(i) Consider a general 2×2 matrix,

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Determine when M^{-1} exists and specify a formula for it (you can look it up or use Mathematica). Then prove that $MM^{-1} = I$.

(j) Find the inverse of the matrices

$$A = \begin{pmatrix} 8 & 1 \\ 2 & 3 \end{pmatrix}, \qquad B = \begin{pmatrix} 3 & 2 \\ 6 & 4 \end{pmatrix},$$

if they exist.

(k) Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be 2 by 2 matrices, where $i, j \in \{1, 2\}$. Prove that

$$(A + B)^T = A^T + B^T, \qquad (AB)^T = B^T A^T.$$

Let A, B be given in (1) and let $C = B + I$. Calculate $(AC)^{-1}$ without calculating AC.

(l) Let A, B be given in (j). Calculate AB and BA.

(m) Let A and B be 3 by 3 invertible matrices with determinants -39 and $\frac{77}{3}$ respectively. Find the determinant of BA.

(n) Let A be an invertible n by n matrix. The trace of a matrix is the sum of the diagonal elements. Prove generally that the trace of $A + B$ is equal to the sum of the traces of A and B.

(o) The eigenvalues of a matrix A are the complex numbers given by

$$\det(A - \lambda I) = 0.$$

Calculate the eigenvalues of the matrix A given in (j).

(p) Consider an $1 \times n$ matrix $A = [1, , \ldots, 1]$ (a row of ones). Calculate the trace of $A'A$ and the trace of AA'.

(q) Consider an $n \times n$ matrix B with elements $b_{i,j} = i + j$. Find a formula for ABA' where A is the row of 1's from the previous problem.

(r) Repeat the previous problem with $b_{i,j} = i + j^2$.

(s) Prove (by counter-example) that $\det(A + B) \neq \det(A) + \det(B)$.