MATH7501 Assignment Solutions

These solutions produced by Mitchell Griggs.

Unit 1

1. Example 6

$$\begin{aligned}
A \cap B &= \{2,4\} \\
A \cup B &= \{1,2,3,4,6,8,10\} \\
B \cup C &= \{1,2,3,4,5,6,7,8,9,10\} \\
B - A &= \{6,8,10\} \\
A - C &= \{2,4\} \\
B^c &= \{1,3,5,7,9\} \\
A^c &= \{5,6,7,8,9,10\} \\
A^c \cup B &= \{2,4,5,6,7,8,9,10\}
\end{aligned}$$

2. Example 8 $y = 3$ and $z = \sqrt{9} = 3$.

3. Example 9

$$A \times B = \{(-1,x),(-1,y),(0,x),(0,y),(1,x),(1,y)\}$$

4. Example 10

(a) F

(b) T

(c) T

(d) F (unless if the universe is $\emptyset$)

(e) T

(f) T

(g) T

(h) F

5. Example 11 No but $\{A_1, \{0\}, A_2\}$ is a partition of $\mathbb{Z}$.

6. Example 12

$$\{\{4n : n \in \mathbb{Z}\}, \{4n+1 : n \in \mathbb{Z}\}, \{4n+2 : n \in \mathbb{Z}\}, \{4n+3 : n \in \mathbb{Z}\}\}$$

7. Example 13

$$\mathcal{P}(B) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, B\}$$

8. $i = e^{\frac{\pi}{2}i} \Rightarrow i^n = e^{\frac{n\pi i}{2}}$.

9. Example 16 (Mathematica.)

10. Example 17(c)

$$\binom{8}{4} = \frac{8!}{4!4!} = 70$$

11. Example 18

$$\binom{5}{3} = \frac{5!}{2!3!} = 10$$

12. Example 19

(a)

$$\binom{10}{3} = 120$$

(b) $10 \times 9 \times 8 = 720$

(c) The first answer counted the number of outcomes when order does not matter. In the second answer, order DOES matter.

13.

$$2^n = (1+1)^n = \sum_{i=0}^{n} \binom{n}{i} 1^i 1^{n-i} = \sum_{i=0}^{n} \binom{n}{i},$$

by the binomial expansion of $(1+1)^n$.

14. Example 21

$$\binom{x+3}{x+1} = \frac{(x+3)!}{(x+1)!2!} = \frac{(x+3)(x+2)}{2} = \binom{x+3}{2}$$

15. The number $\binom{n+1}{r}$ is the number of *r-subsets* (an *r-subset* is a subset with cardinality r) of a set with cardinality $n+1$.

Another way to count this number is to consider an element from the $(n+1)$-set. Call this element x. Out of each subset of the $(n+1)$-set, if x is in the subset then there are $\binom{n}{r-1}$ way to choose the other elements, but if x is not in the subset then there are $\binom{n}{r}$ ways to choose the subset. Thus,

$$\binom{n+1}{r} = \binom{n}{r-1} + \binom{n}{r}.$$

16. Example 22

(a) 28

(b) 210

(c) 10

(d) 21

17.

$\mathbb{Z}^+$	$\mathbb{Q}$
0	0
1	1
2	-1
3	$2 = 2/1$
4	$-2 = -2/1$
5	$1/2$
6	$-1/2$
7	$3 = 3/1$
8	$-3 = -3/1$
9	$3/2$
$\vdots$	$\vdots$

Can you see a pattern? We can label each rational number. This is a version of *Cantor's Diagonal Argument.*

Unit 2

18. Example 30

(i) $(1,0),(2,0),(2,2),(3,0),(3,2),(4,0),(4,2),(4,4),(5,0),(5,2),(5,4)$

(ii) $(2,2),(4,4)$

(iii) $(2,0),(2,2),(2,4),(2,6),(2,8),(4,0),(4,2),(4,4),(4,6),(4,8)$

(iv) $(1,6),(3,4),(5,2)$

(v) $(2,8),(3,8),(4,6),(4,8),(5,6),(5,8)$

19. Example 31 Examples:

$x\rho y$ if and only if $|x| \geqslant y$

$x\rho y$ if and only if $|x| < y-1$

$x\rho y$ if and only if $x^2 = y$

$x\rho y$ if and only if $x = y+2$

$\vdots$

20. Example 32

(i) $(0,0),(-1,0),(1,5),\ldots$

(ii) $(1,0),(0,-2),(100,2),\ldots$

(iii) $(0,0),(1,1),(1,-1),\ldots$

(iv) $(0,0),(7,7),(-4,-4),\ldots$

(v) $(0,-1),(5,4),(-2,-3),\ldots$

(vi) $(0,0),(1,2),(-5,8),\ldots$

21. Example 34 See Figure 1.

22. Example 35 No; $(6,2) \in R$ and $(6,8) \in R$ but $2 \neq 8$.

23. Example 36 $\rho^{-1} = \{(4,x),(10,x),(1,z),(7,y),(1,y)\}$

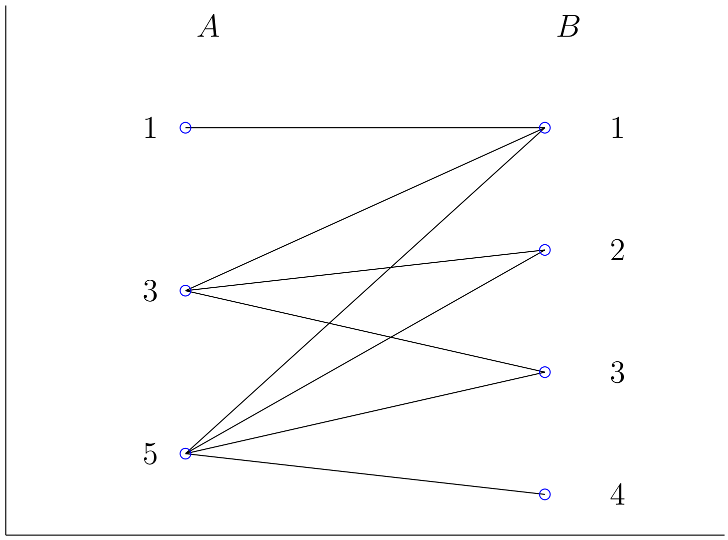

Figure 1

24. Example 37

(a) R is the set whose elements are the following pairs: $(1,1),(1,4),(1,7),$
$(1,10),(2,5),(2,8),(2,2),(3,3),(3,6),(3,9),(4,1),(4,4),(4,7),(4,10),$
$(5,2),(5,5),(5,8),(6,3),(6,6),(6,9),(7,1),(7,4),(7,7),(7,10),(8,2),$
$(8,5),(8,8),(9,3),(9,6),(9,9),(10,1),(10,4),(10,7),(10,10)$

(b) $R^{-1} = R$ because $3|(x-y)$ if and only if $3|(y-x)$.

(c) See Figure 2.

25. Example 38

R_1 is not reflexive, is not symmetric, and is not transitive.

R_2 is reflexive, is symmetric, and is transitive.

R_3 is reflexive, is symmetric, and is transitive.

R_4 is not reflexive, is not symmetric, and is not transitive.

R_5 is reflexive, is not symmetric, and is transitive.

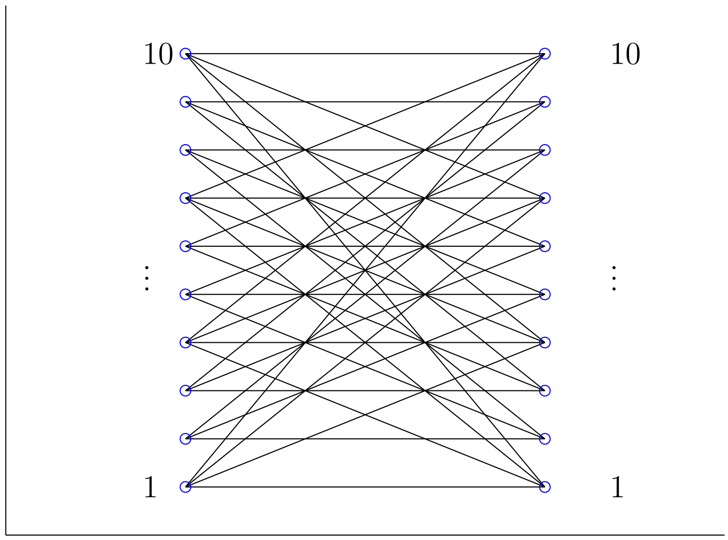

Figure 2

R_6 is reflexive, is symmetric, and is transitive.

26. Example 41

(a) σ is not reflexive.

(b) σ is not symmetric.

(c) σ is transitive.

27. Example 43

(i) ρ is reflexive.

(ii) ρ is symmetric.

(iii) ρ is transitive.

28. Example 44 If R is symmetric then $(x, y) \in R \Rightarrow (y, x) \in R$, and $R^{-1} = \{(y, x) : (x, y) \in R\}$, so $R = R^{-1}$. Conversely, if $(x, y) \in R$ and $R = R^{-1}$, then $(y, x) \in R^{-1} = R$ so $(y, x) \in R$, showing that R is symmetric.

29. Example 45 True; if R is transitive then $(x, y), (y, z) \in R \Rightarrow (x, z) \in R$.

If $(a, b), (b, c) \in R^{-1}$ then $(b, a), (c, b) \in R$, which means $(c, b), (b, a) \in R$, so $(c, a) \in R$ (by the transitivity of R) and therefore $(a, c) \in R^{-1}$. In other words, $(a, b), (b, c) \in R^{-1} \Rightarrow (a, c) \in R^{-1}$, showing that R^{-1} is transitive.

30. Example 46

$R = \{(1,1),(1,5),(1,9),(5,1),(5,5),(5,9),(3,3),(3,7),(7,3),(7,7),(11,11)\}$

31. Example 47 See Figure 3. R is an equivalence relation.

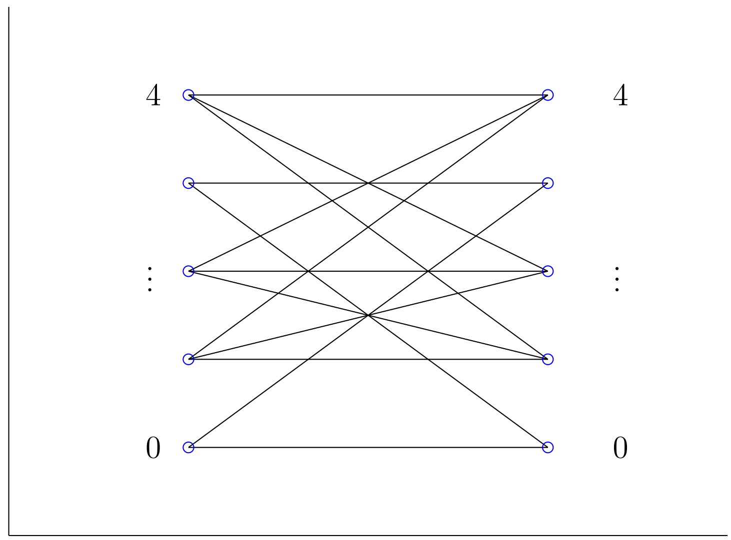

Figure 3

32. Example 51 S is not a function because $(0, 0) \in S$ and $(0, 4) \in S$ but $0 \neq 4$.

33. Example 52 f is not a function because (for example)

$$f(1/2) = 1 \neq 2 = f(2/4),$$

but $1/2 = 2/4$. In other words, $(1/2, 1) \in f$ and $(2/4, 2) = (1/2, 2) \in f$ but $1 \neq 2$.

34. Example 54

α is not a function since $\alpha(2)$ is not defined.

β is a function.

γ is not a function; $(1, a), (1, b) \in \gamma$ but $a \neq b$.

δ is not a function; $\delta(3)$ is not defined.

35. Example 55 For every $x \in \mathbb{R}$,

$$f(x) = x = \sqrt[3]{x^3} = g(x),$$

so $f = g$.

36. Example 56 $f(-1) = -1 \neq 1 = g(-1)$ so $f \neq g$.

37. Example 59

$$(3, 4, 5) \star (5, 12, 13) = (-33, 45, 65)$$

38. $e^{i\theta} = \cos\theta + i\sin\theta$. This result gives formulae such as

$$\begin{aligned}(\cos\theta + i\sin\theta)^n &= \left(e^{i\theta}\right)^n \\ &= e^{i(\theta n)} = \cos(n\theta) + i\sin(n\theta),\end{aligned}$$

which is *de Moivre's Theorem*. For example, if $n = 2$ then this gives

$$\begin{aligned}\cos(2\theta) &= \cos^2\theta - \sin^2\theta \quad \text{and} \\ \sin(2\theta) &= 2\cos\theta\sin\theta.\end{aligned}$$

39. Example 62

(a)

$$\begin{aligned}(f \circ g)(x) &= f(g(x)) = (2x - x^2)^3, \\ (g \circ f)(x) &= g(f(x)) = 2x^3 - x^6.\end{aligned}$$

(b) No; $f \circ g \neq g \circ f$.

40. Example 63

(a) See Figure 4.

(b) $\{y, z\}$

41. Example 65

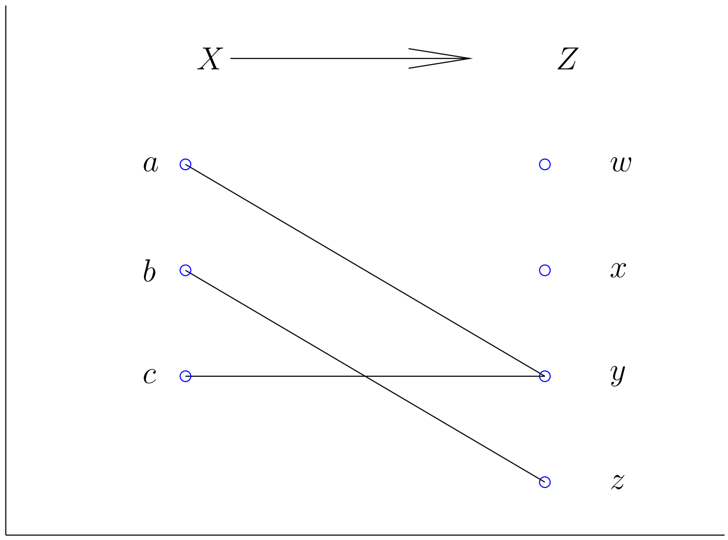

Figure 4

(a) See Figure 5.

(b) $s \circ t$ is onto.

42. Example 67

(a) f is not one-to-one; $f(-1) = 2 = f(1)$.

(b) g is injective; if $x_1, x_2 \in \mathbb{R}$ and $g(x_1) = g(x_2)$ then

$$2x_z^3 - 1 = 2x_2^3 - 1 \Rightarrow x_1 = x_2.$$

43. Example 68

(a) **(i)** 32

(ii) 28

(iii) 32

(iv) 98

(b) H is not one-to-one; $H(40076832) = H(41134032)$ but $40076832 \neq 41134032$.

44. Example 70

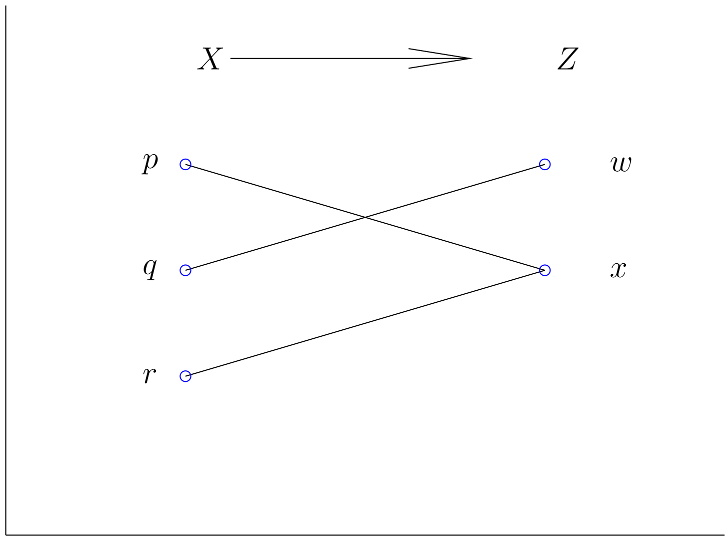

Figure 5

(a) f is surjective; for any $y \in \mathbb{R}^+ \cup \{0\}$, the number $-\sqrt{y} \in \mathbb{R}$ satisfies $f(-\sqrt{y}) = y$.

(b) g is not onto; if $g(x) = -1$ then x is not in the domain ($\mathbb{Z}$) of g.

45. Example 72 f is one-to-one.

46. Example 73 If $f(x_1) = f(x_2)$ then we find $x_1 = x_2$, so f is one-to-one.

47. Example 74 f^{-1} does not exist because f is not one-to-one. g^{-1} exists and is described by the diagram of Figure 6.

48. Example 75 Say $y = g(x) = 2x + 5$, so

$$g^{-1}(y) = x = \frac{y-5}{2}.$$

Therefore $g^{-1}(x) = \frac{x-5}{2}$.

50. $\arcsin : (-1, 1) \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$; see Figure 7.

51. Example 78 See Figure 8.

52. Example 79 If $z = ax + by + d$ then $(0, 0, 5)$ gives $d = 5$ and then $(0, 1, 1)$ is used to find $b = -4$. Next, $(1, 3, 2)$ yields $a = 9$, so the equation of the plane is

$$z = 9x - 4y + 5.$$

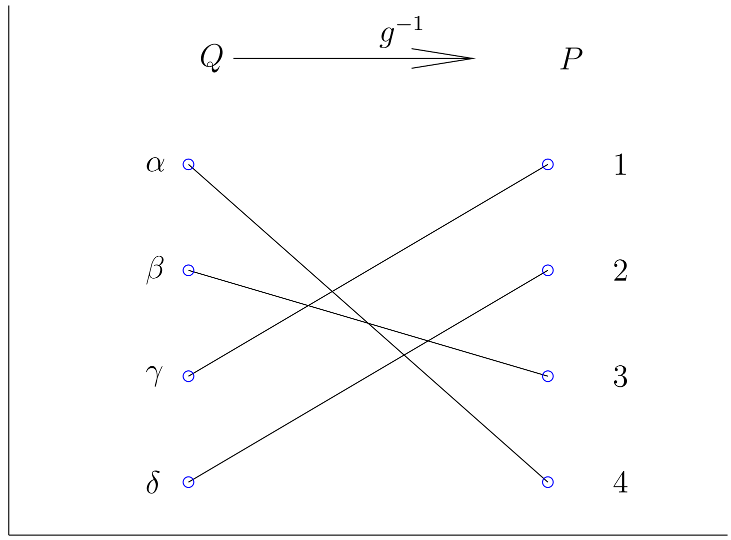

Figure 6

53. Example 80 See Figure 9.

54. Example 81 See Figure 10.

55. Example 82 See Figure 11.

56. Example 83 Say $z = mx+ny+c$. We have $\Delta z = 1-0 = 1$. Considering when $y = 0$,

$$0 = -2x + 5 \Rightarrow x = 5/2,$$
$$0 = -2x + 2 \Rightarrow x = 1,$$

so $\Delta x = 1 - 5/2 = -3/2$, giving $m = \frac{\Delta z}{\Delta x} = \frac{-2}{3}$.

Similarly, from considering $x = 0$, we get $\Delta y = 2 - 5 = -3 \Rightarrow n = \frac{-1}{3}$.

Therefore $z = \frac{-2}{3}x - \frac{1}{3}y + c$. When $z = 0$, we then get $c = 5/3$, so the equation of the plane is

$$z = \frac{-2}{3}x - \frac{1}{3}y + \frac{5}{3}.$$

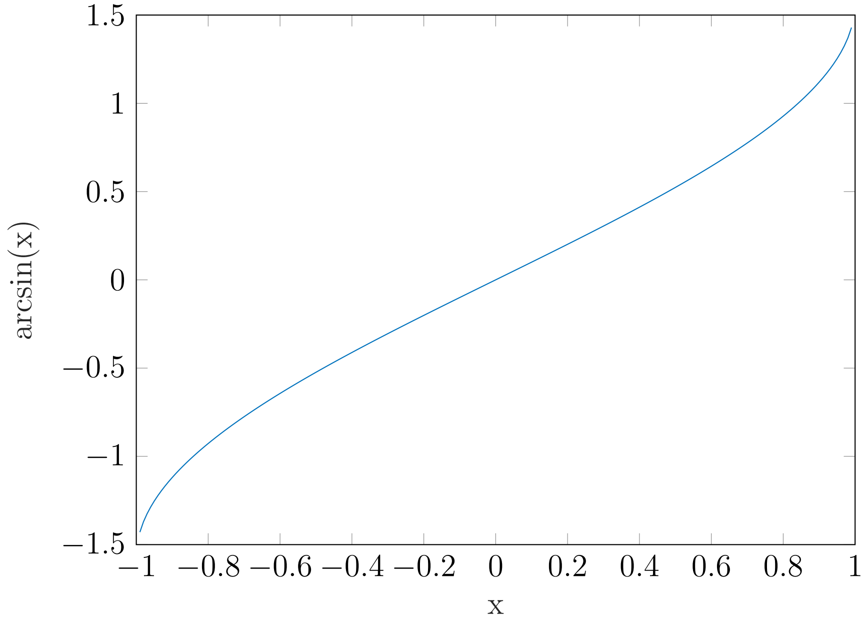

Figure 7

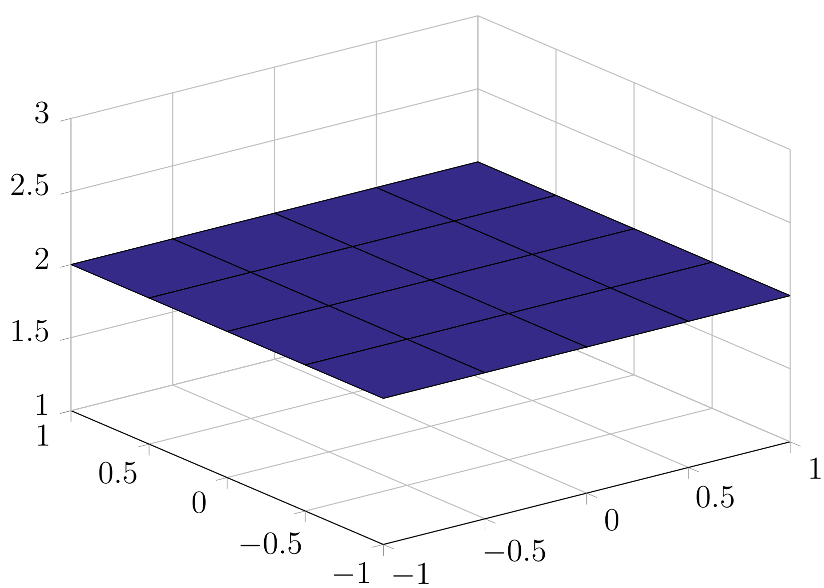

Figure 8

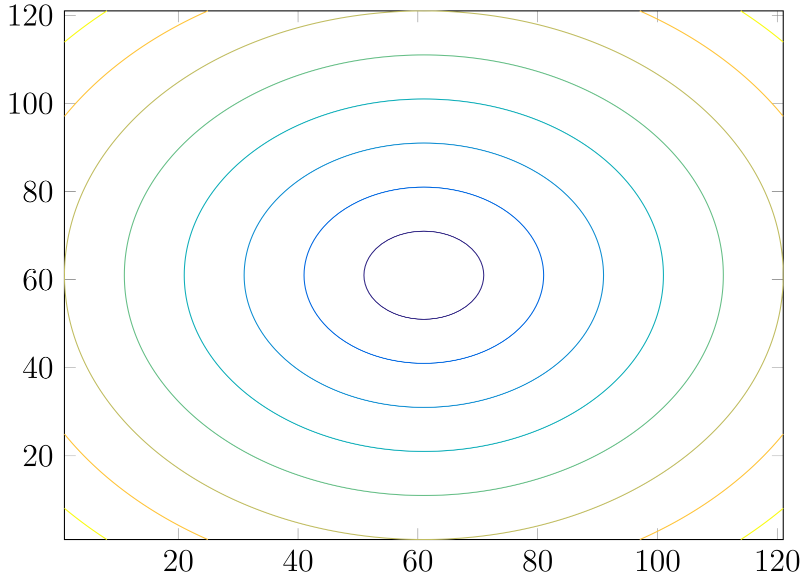

Figure 9

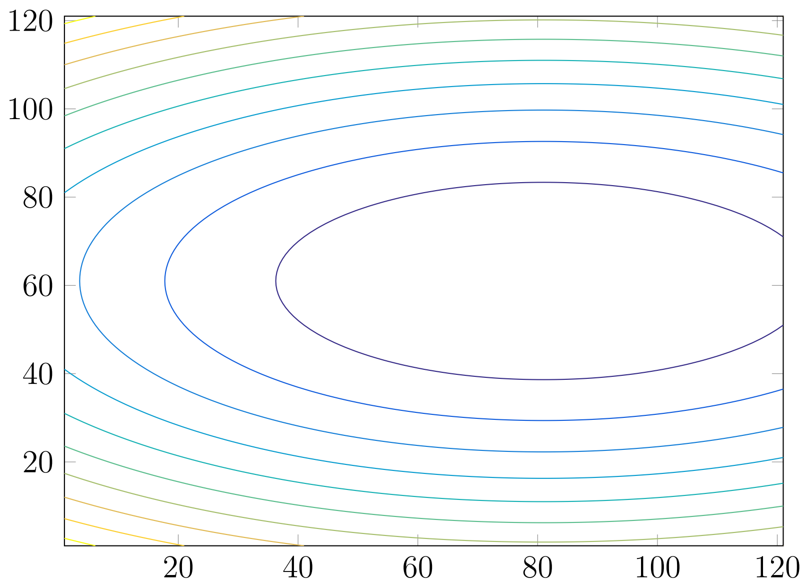

Figure 10

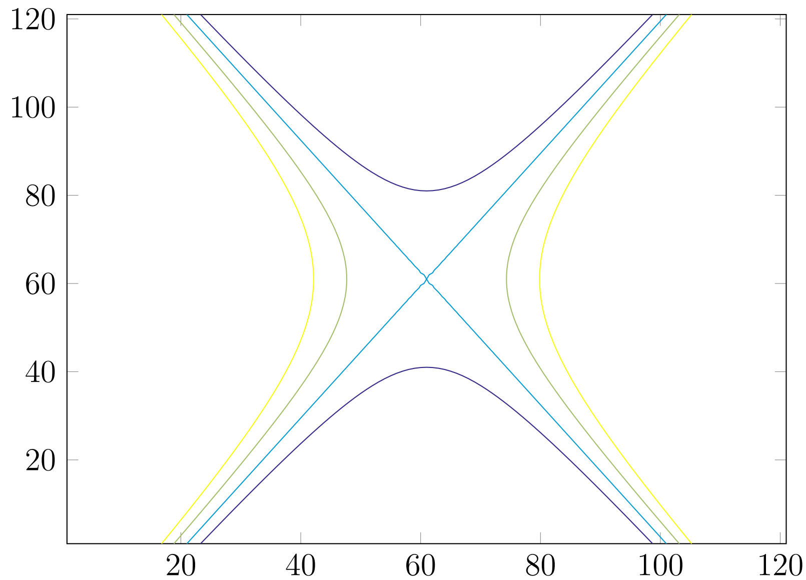

Figure 11