MATH7501 Assignment Solutions

These solutions produced by Mitchell Griggs.

Unit 3

1. Example 89 If $n = 6$ then $1+2+3+4+5 = 15$ but there is no integer k where $6k = 15$.

2. Example 90

(a) T

(b) T

(c) F

3. Example 91 If a and b are rational numbers then $a = \frac{m}{n}$ and $b = \frac{p}{q}$, where $m, n, p, q \in \mathbb{Z}$, $n \neq 0$, and $q \neq 0$. The product

$$ab = \frac{mp}{nq}$$

is also rational, since $mp, nq \in \mathbb{Z}$ and $nq \neq 0$.

4. Example 92 Writing $r = \frac{m}{n}$ for $m, n \in \mathbb{Z}$ and $n \neq 0$, let $s = \frac{-m}{n}$. Now conclude that $r + s = 0$ and $s + r = 0$.

5. Example 94 Writing $r = \frac{m}{n}$ for $m, n \in \mathbb{Z} - \{0\}$, define $r^{-1} = \frac{n}{m}$. Conclude that $r \cdot r^{-1} = 1$ and $r^{-1} \cdot r = 1$.

6. Example 95

(a) T; $72 = 4k$ for some integer k.

(b) F; there is no $k \in \mathbb{Z}$ where $24 = 48k$.

(c) F; there is no $k \in \mathbb{Z}$ where $5 = 0k$.

(d) T; $9 = (-3) \times (-3)$.

7. Yes; for any $a, b \in \mathbb{Z}$, $2a(3b+3) = 6(ab+a)$.

8. No; if $a = 1$ and $b = 0$ then $2a(4b+1) = 2$.

9. Example 96

(a) $5440 = 2^6 \times 5 \times 17$

(b) $43560 = 2^3 \times 3^2 \times 5 \times 11^2$

10. There exist $m, n \in \mathbb{Z}$ such that $a = km$ and $b = kn$, so

$$a + b = k(m + n),$$

giving $k|(a + b)$ since $m + n \in \mathbb{Z}$.

11. Example 97

(a) F

(b) T

(c) F

12. Example 98

(a) $102 = 11 \times 9 + 3$

(b) $-4 = 5 \times (-1) + 1$

13. Example 99

$$\begin{aligned} ab &= (4x + 1)(4y + 1) \\ &= 16xy + 4x + 4y + 1 \\ &= 4m + 1, \end{aligned}$$

where $m = 4xy + x + 4$.

14. Example 100 By definition,

$$u = d \cdot p + v \ \text{ and } \ w = d \cdot q + x,$$

for some $p, q \in \mathbb{Z}$. Therefore

(a) $u + w = d \cdot (p + q) + (v + x)$, so $u + w \equiv v + x \pmod{d}$, and

(b) $uw = d \cdot (dpq + px + qv) + xv$, so $uw \equiv xv \pmod{d}$.

15. Example 101 Suppose that x is an odd integer, so $x = 2y + 1$ for some $y \in \mathbb{Z}$. Now

$$x^2 = 4y(y + 1) + 1.$$

One of y and $y+1$ is even, so there is some $z \in \mathbb{Z}$ such that $y(y+1) = 2z$. Therefore

$$x^2 = 4 \cdot 2z + 1 = 8m + 1,$$

where $m = z$.

16. Example 104

(a) For every x in the set of squares, x is a rectangle.

(b) Every integer is a real number.

17. Example 106

(a) $\exists\, x \in \mathbb{R}$ such that $x \in \mathbb{Q}$.

(b) There is an elephant that is white.

18. Example 108(b)

$$\forall\, x \in \mathbb{R},\ \text{if } x \in \mathbb{Z} \text{ then } x \in \mathbb{Q}.$$

19. Example 111 Suppose that $r = \frac{a}{b}$ is a nonzero, rational number, so $a, b \in \mathbb{Z} - \{0\}$, and suppose that $s \in \mathbb{R} - \mathbb{Q}$. Towards a contradiction, suppose $rs \in \mathbb{Q}$. Therefore $rs = \frac{m}{n}$ for some $n \in \mathbb{Z} - \{0\}$ and $m \in \mathbb{Z}$, but $r \neq 0$ and $s \neq 0$ so $m \neq 0$.

Now $s = \frac{m}{nr} = \frac{mb}{na}$, but $mb \in \mathbb{Z}$ and $na \in \mathbb{Z} - \{0\}$, so $s \in \mathbb{Q}$. This contradiction concludes our proof.

20. Example 112 Suppose that $p > 3$ is prime. Towards a contradiction, suppose that p is not $\pm 1 \pmod 6$. Therefore p is 0,2,4, or 3 (mod 6).

If p is 0, 2, or 4 (mod 6), then it is even (and greater than 3) so can't be prime. The only remaining possibility is $p \equiv 3 \pmod 6$, so $p = 6k + 3$ for some $k \in \mathbb{Z}$. However, $p > 3$ so $k \geqslant 1$ (meaning that $2k + 1 > 1$), and therefore $p = 3(2k+1)$ is not prime.

21. We want to show that

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}, \tag{1}$$

for every $n \in \mathbb{N}$.

When $n = 1$,

$$\sum_{i=1}^{n} i^2 = 1 \text{ and } \frac{n(n+1)(2n+1)}{6} = \frac{1 \cdot 2 \cdot 3}{6} = 1,$$

so the formula holds when $n = 1$.

Assume that Equation (1) holds when $n = k \in \mathbb{N}$.

If we now consider the next value, $n = k + 1$, then

$$\begin{aligned}
\sum_{i=1}^{k+1} i^2 &= (k+1)^2 + \sum_{i=1}^{k} i^2 = (k+1)^2 + \frac{k(k+1)(2k+1)}{6} \\
&= (k+1)\left[\frac{6k+6+k(2k+1)}{6}\right] = (k+1)\left[\frac{4k+6+k(2k+3)}{6}\right] \\
&= (k+1)\left[\frac{2(2k+3)+k(2k+3)}{6}\right] = (k+1)\left[\frac{(k+2)(2k+3)}{6}\right] \\
&= \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6} = \frac{n(n+1)(2n+1)}{6},
\end{aligned}$$

so Equation (1) holds when $n = k + 1$. We have therefore proved Equation (1), for every $n \in \mathbb{N}$, with mathematical induction.

22. If $n = 1$ then $1 + 2^{2n-1} = 1 + 2^1 = 3$, which is divisible by 3. Now suppose that

$$3 | \left(1 + 2^{2n-1}\right),$$

so $1 + 2^{2n-1} = 3k$ for some $k \in \mathbb{Z}$. Then

$$\begin{aligned}
1 + 2^{2(n+1)-1} &= 1 + 2^{2n-1+2} = 1 + 4 \cdot \left(2^{2n-1} + 1 - 1\right) \\
&= 1 + 4(3k-1) = 1 + 12k - 4 = 12k - 3,
\end{aligned}$$

which is divisible by 3. We have shown the result, by mathematical induction.

23. Example 113 Consider the equation

$$\sum_{i=1}^{n} (2i-1) = n^2. \tag{2}$$

When $n = 1$, each side of Equation (2) is 1. Now suppose that Equation (2) holds for some $n \in \mathbb{N}$. From this assumption, it follows that

$$\sum_{i=1}^{n+1} (2i-1) = [2(n+1)-1] + n^2 = 2n + 1 + n^2 = (n+1)^2,$$

so Equation (2) holds for all $n \in \mathbb{N}$, by mathematical induction.

24. Example 114 Consider the equation

$$\sum_{j=1}^{n} \frac{1}{j(j+1)} = \frac{n}{n+1}. \tag{3}$$

When $n = 1$, the equation holds, with each side being $1/2$. Now assume that Equation (3) holds for some $n \in \mathbb{N}$. We then have

$$\begin{aligned}\sum_{j=1}^{n+1} \frac{1}{j(j+1)} &= \frac{1}{(n+1)(n+2)} + \sum_{j=1}^{n} \frac{1}{j(j+1)} = \frac{1}{(n+1)(n+2)} + \frac{n}{(n+1)} \\ &= \frac{1+n(n+2)}{(n+1)(n+2)} = \frac{1+n^2+2n}{(n+1)(n+2)} = \frac{n+1}{n+2},\end{aligned}$$

so Equation (3) holds for all $n \in \mathbb{N}$, by mathematical induction.

25. Example 115 When $t = 1$,

$$\sum_{j=1}^{t} 2^{j-1} = 2^0 = 1 = 2^1 - 1 = 2^t - 1.$$

Supposing

$$\sum_{j=1}^{t} 2^{j-1} = 2^t - 1,$$

then

$$\sum_{j=1}^{t+1} 2^j = 2^{(t+1)-1} + \sum_{j=1}^{t} 2^{j-1} = 2^t + 2^t - 1 = 2 \cdot 2^t - 1 = 2^{t+1} - 1,$$

showing the result, by mathematical induction.

26. See Table 1.

27. See Table 2. The statement is a contradiction.

26. See Table 3 for commutativity and Table 4 for associativity.

29. Example 116 If you study then you pass the test.

30. Example 117 If you passed the test then you studied.

p	q	$(p \vee q)$	$\wedge$	$(\sim p)$
T	T	T	F	F
T	F	T	F	F
F	T	T	T	T
F	F	F	F	T
			$\uparrow$	

Table 1

p	q	$(p \wedge q)$	$\vee$	$((\sim p) \vee (p \wedge (\sim q)))$
T	T	T	F	F
T	F	F	F	T
F	T	F	F	F
F	F	F	F	F
			$\uparrow$	

Table 2

p	q	$(p \wedge q)$	$\leftrightarrow$	$(q \wedge p)$	$(p \vee q)$	$\leftrightarrow$	$(q \vee p)$
T	T	T	T	T	T	T	T
T	F	F	T	F	T	T	T
F	T	F	T	F	T	T	T
F	F	F	T	F	F	T	F
			$\uparrow$			$\uparrow$	

Table 3

p	q	r	$(p \wedge q) \wedge r$	$\leftrightarrow$	$q \wedge (p \wedge r)$	$(p \vee q) \vee r$	$\leftrightarrow$	$p \vee (q \vee r)$
T	T	T	T	T	T	T	T	T
T	T	F	F	T	F	T	T	T
T	F	T	F	T	F	T	T	T
T	F	F	F	T	F	T	T	T
F	T	T	F	T	F	T	T	T
F	T	F	F	T	F	T	T	T
F	F	T	F	T	F	T	T	T
F	F	F	F	T	F	F	T	F
				$\uparrow$			$\uparrow$	

Table 4

p	q	p	$\Rightarrow$	$(q \wedge (\sim p))$
T	T	T	F	F
T	F	T	F	F
F	T	F	T	T
F	F	F	T	F
			$\uparrow$	

Table 5

31. Example 118 See Table 5.

32. Suppose that we have some Boolean formula of interest. The *Boolean Satisfiability Problem is*

> "Can we find truth values (for all variables) so that the formula's result is true?"

In other words, is the formula *satisfiable*?

As an example, $p \wedge (\sim p)$ is *unsatisfiable*, since it is always false. An example of a formula that is satisfiable is $(p \vee q) \wedge (\sim p)$. This is showb by considering when p is false and q is true.