MATH7501 Assignment Solutions

These solutions produced by Mitchell Griggs.

Unit 4

1. Example 119 $a_i = (i^i)(-1)^i = (-i)^i$, $i = 1, 2, 3, \ldots$, is one such sequence.

2. Example 120

$$\sum_{i=1}^{n} \frac{1}{i(i+1)}$$

3. Example 124

$$\begin{aligned} F_8 &= \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^8 - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^8 \\ &= \frac{1}{\sqrt{5}}\frac{1}{2^8}\left[\sum_{i=0}^{8}\binom{8}{i}\left(\left(\sqrt{5}\right)^i - \left(-\sqrt{5}\right)^i\right)\right] \\ &= \frac{1}{16\sqrt{5}}\left[\sqrt{5} + 7\left(\sqrt{5}\right)^3 + 7\left(\sqrt{5}\right)^5 + \left(\sqrt{5}\right)^7\right] \\ &= \frac{336}{16} = 21, \end{aligned}$$

which agrees with Example 123.

4. For any $n \in \mathbb{N}$, if $a_n = 5 \cdot 2^n$ then it follows that

$$a_n = 5 \cdot 2^{n-1} \cdot 2 = 2 \cdot \left(5 \cdot 2^{n-1}\right) = 2a_{n-1}.$$

5. The *Catalan Numbers* are $C_0, C_1, C_2, \ldots$, where

$$C_n = \frac{1}{n+1}\binom{2n}{n},$$

for each $n = 0, 1, 2, \ldots$.

These numbers are characterised in many equivalent ways. For example, C_n is the number of ways that n pairs of parentheses can be correctly ordered (with one open bracket occurring for each closed bracket, and occurring *before* that closed bracket).

With this interpretation, suppose that we know $C_0, C_1, \ldots, C_n$, and we have n pairs of parentheses. If we now need to insert a new $(n+1)th$ pair, then we counting how this may be done leads to the formula

$$C_{n+1} = \sum_{i=0}^{n} C_i C_{n-i}.$$

Other formulae are possible.

6. Example 125(4) $a_n = r^n$ for each $n = 0, 1, 2, \ldots$.

(1) When $r = 1/2$

(2) When $r = 1$, $\lim_{n\to\infty} a_n = 1$.

(3) When $r = 2$, $\{a_n\}$ diverges.

7. Example 126 $a_n = \sin(\log(n))/n$ for each $n = 1, 2, 3, \ldots$.

For each $n \geqslant 1$,

$$-1 \leqslant \sin(\log(n)) \leqslant 1$$

, so

$$\frac{-1}{n} \leqslant \frac{\sin(\log(n))}{n} \leqslant \frac{1}{n},$$

and both $\frac{-1}{n} \to 0$ and $\frac{1}{n} \to 0$ as $n \to \infty$, so $\lim_{n\to\infty} a_n = 0$, by the Squeeze Theorem.

8. One way to show why $\lim_{n\to\infty} n^{1/n} = 1$ is by writing

$$n^{1/n} = \exp(\log(n)/n) = e^{\log(n)/n},$$

and knowing that you can apply the following steps:

$$\begin{aligned}\lim_{n\to\infty} n^{1/n} &= \lim_{n\to\infty} \left(\exp\left(\frac{\log(n)}{n}\right)\right) \\ &= \exp\left(\lim_{n\to\infty} \frac{\log(n)}{n}\right) = \exp(0) = 1.\end{aligned}$$

9. Example 128 Consider the partial sums. That is,

$$\sum_{n=0}^{\infty} \frac{1}{(n+1)(n+2)} = \lim_{n\to\infty} S_n,$$

where

$$S_n = \sum_{k=0}^{n-1} \frac{1}{(k+1)(k+2)} = \sum_{k=1}^{n} \frac{1}{k(k+1)}.$$

In Assignment 2, you may have shown (with mathematical induction) that

$$\sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{n}{n+1},$$

so

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \left(\frac{n}{n+1}\right) = \lim_{n\to\infty} \left(1 - \frac{1}{(n+1)}\right) = 1.$$

10. The *Harmonic Series* is

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

We can write this as

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{n} &= 1 + \frac{1}{2} + \sum_{n=3}^{4} \frac{1}{n} + \sum_{n=5}^{8} \frac{1}{n} + \sum_{n=9}^{16} \frac{1}{n} + \cdots \\ &\geqslant 1 + \frac{1}{2} + 2 \cdot \frac{1}{4} + 4 \cdot \frac{1}{8} + 8 \cdot \frac{1}{16} + \cdots \\ &= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots,\end{aligned}$$

so the series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, by the comparison test.

11. Example 129 with 7%

$$\sum_{n=1}^{\infty} 100(0.93)^n = \frac{100}{1-0.93} \approx 1428.57.$$

12.a This limit is zero.

12.b $\lim_{n\to\infty}(\pi/4)^n = 0$ since $|\pi/4| < 1$.

12.c

$$\lim_{n\to\infty}\frac{n^3+3}{n^3+n^2-1}=\lim_{n\to\infty}\frac{\frac{1}{n}+\frac{3}{n^3}}{1+\frac{1}{n}-\frac{1}{n^3}}=\frac{0}{1}=0.$$

12.d This solution uses Taylor Series:

$$n\sin\frac{\pi}{n}=n\cdot\left(\frac{\pi}{n}-\frac{(\pi/n)^3}{3!}+\frac{(\pi/n)^5}{5!}-\cdots\right)\to\pi$$

as $n\to\infty$.

12.e

$$\frac{1}{n}-\frac{1}{(n+1)}=\frac{n+1-n}{n^2+n}=\frac{1}{n^2+n}\to 0$$

as $n\to\infty$.

12.f

$$\lim_{n\to\infty}\left(\sqrt{n+1}-\sqrt{n+2}\right)=\lim_{n\to\infty}\left(\sqrt{n}-\sqrt{n+1}\right),$$

but

$$\sqrt{n}-\sqrt{n+1}=\left(\sqrt{n}-\sqrt{n+1}\right)\frac{\left(\sqrt{n}+\sqrt{n+1}\right)}{\left(\sqrt{n}+\sqrt{n+1}\right)}=\frac{n-n-1}{\sqrt{n}+\sqrt{n+1}}$$

$$=\frac{-1}{\sqrt{n}+\sqrt{n+1}}\geqslant\frac{-1}{2\sqrt{n+1}}\to 0,$$

so $\sqrt{n}-\sqrt{n+1}\to 0$ by the Squeeze Theorem and the Comparison Test.

12.g $\cos^2(n\pi)=1$ for all $n\in\mathbb{N}$ and so

$$\sum_{n=1}^{\infty}\frac{\cos^2(n\pi)}{n!}=\sum_{n=1}^{\infty}\frac{1}{n!}=\sum_{n=1}^{\infty}\frac{1^n}{n!}=e^1.$$

12.h

$$\sum_{n=2}^{\infty}\left(\frac{1}{n-1}-\frac{1}{n+1}\right)=\sum_{n=2}^{\infty}\frac{(n+1)-(n-1)}{n^2-1}$$

$$=\sum_{n=2}^{\infty}\frac{2}{n^2-1}=2\sum_{n=2}^{\infty}\frac{1}{n^2-1},$$

which converges by the Comparison Test with a p-series (when $p = 2$).

12.i

$$\sum_{n=1}^{\infty} \frac{n + \sin n}{n^4 + n} = \sum_{n=1}^{\infty} \frac{1}{n^3 + 1} + \sum_{n=1}^{\infty} \frac{\sin n}{n^4 + n},$$

which are both convergent series;

$$\sum_{n=1}^{\infty} \frac{1}{n^3 + 1} \leqslant \sum_{n=1}^{\infty} \frac{1}{n^3}$$

converges (p-series) and

$$\sum_{n=1}^{\infty} \frac{\sin n}{n^4 + n}$$

(absolutely) converges (Comparison Test with a p-series), so

$$\sum_{n=1}^{\infty} \frac{n + \sin n}{n^4 + n}$$

converges.

12.j $\lim_{n\to\infty} \frac{n^n}{n!} = \infty$ so

$$\sum_{n=1}^{\infty} \frac{n^n}{n!}$$

diverges since its terms don't approach zero.

12.k The outer circle has diameter 1, so the diagonals of the outer circle are both 1, giving the side length of the inner square as $\sqrt{1/2}$, which is also the diameter of the inner circle, and so on. The sum of the areas of the squares is

$$1^2 + \left(\sqrt{1/2}\right)^2 + \left(\sqrt{1/4}\right)^2 + \left(\sqrt{1/8}\right)^2 + \cdots = \sum_{i=0}^{\infty} \frac{1}{2^i} = 2.$$

The sum of the areas of the circles is

$$\pi \left(1/2\right)^2 + \pi \left(\frac{1}{2}\sqrt{1/2}\right)^2 + \cdots = \sum_{i=1}^{\infty} \pi \left(\frac{1}{2}\left(\frac{1}{2^i}\right)\right)^2 = \frac{\pi}{2}.$$

The difference is

$$2 - \frac{\pi}{2} = \frac{4 - \pi}{2} \approx 0.43.$$

12.l By the Ratio Test,

$$\lim_{n\to\infty} \left|F_{n+1}^{-1} \div F_n^{-1}\right| = \lim_{n\to\infty} \left|\frac{F_n}{F_{n+1}}\right| = \frac{1}{\varphi}$$

(where $\varphi = \frac{1+\sqrt{5}}{2}$), since $\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \varphi$. As $\frac{1}{\varphi} < 1$,

$$\sum_{n=0}^{\infty} F_n^{-1}$$

converges by the Ratio Test.

12.m

$$\begin{aligned}
\lim_{n\to\infty} P(k; n, p_n) &= \lim_{n\to\infty} \binom{n}{k} p_n^k (1-p_n)^{n-k} \\
&= \lim_{n\to\infty} \frac{n!}{(n-k)!k!} \left(\frac{\lambda}{n}\right)^k \left(1 - \frac{\lambda}{n}\right)^{n-k} \\
&= \lim_{n\to\infty} \frac{n(n-1)\cdots(n-k)(n-k-1)\cdots 1}{n^k} \frac{\lambda^k}{k!}(1-\lambda/n)^{n-k} \\
&= \lim_{n\to\infty} \left(\frac{n^k + [\text{lower-order terms}]}{n^k}\right) \frac{\lambda^k}{k!}(1-\lambda/n)^{n-k} \\
&= 1 \cdot \frac{\lambda^k}{k!} \lim_{n\to\infty} (1-\lambda/n)^{n-k} \\
&= \frac{\lambda^k}{k!} \lim_{n\to\infty} (1-\lambda/n)^{-k} \lim_{n\to\infty} (1-\lambda/n)^n \\
&= \frac{\lambda^k}{k!} e^{-\lambda}.
\end{aligned}$$

13. The function f is as described, defined for $x = 0, 1, \ldots, n_1$. As $n_1 = pn_3$, and $p \in (0, 1)$, it follows that when $n_3 \to \infty$ then $n_1 \to \infty$.