MATH7501 Assignment Solutions

These solutions produced by Mitchell Griggs.

(a)

$$\lim_{x\to 0^+}\left(\frac{1}{x}-\frac{1}{\sin x}\right)=\lim_{x\to 0^+}\left(\frac{\sin(x)-x}{x\sin x}\right)$$
$$=\lim_{x\to 0^+}\frac{\frac{-x^3}{3!}+\frac{x^5}{5!}-\cdots}{x^2-\frac{x^4}{3!}+\frac{x^6}{5!}-\cdots}=0.$$

(b)

$$\lim_{x\to\pi/4}(1-\tan(x))(\sec(2x))=\lim_{x\to\pi/4}\frac{\cos(x)-\sin(x)}{\cos(x)\cos(2x)}$$
$$=\lim_{x\to\pi/4}\left(\frac{1}{\cos(x)}\right)\lim_{x\to\pi/4}\frac{\cos(x)-\sin(x)}{\cos(2x)}$$
$$=\sqrt{2}\lim_{x\to\pi/4}\frac{-\sin(x)-\cos(x)}{-2\sin(2x)}=\sqrt{2}\cdot\frac{1}{\sqrt{2}}=1.$$

(c)

$$\lim_{x\to\infty}x\left(2^{1/x}-1\right)=\lim_{t\to 0^+}\frac{1}{t}\left(2^t-1\right)$$
$$=\lim_{t\to 0^+}\frac{2^t\log(2)}{1}=\log(2).$$

(d)

$$\lim_{x\to 1}x^{\frac{1}{1-x}}=\lim_{x\to 1}\exp\left(\frac{1}{1-x}\log(x)\right)$$
$$=\exp\left(\lim_{x\to 1}\frac{\log(x)}{1-x}\right)=\exp\lim_{x\to 1}\left(\frac{1}{x\cdot(-1)}\right)=e^{-1}.$$

(e) For continuity, we need

$$2(1)-(1)^2=(1)^2+k(1)+p,$$

which rearranges to give

$$k=-p.$$

For differentiability, we also need

$$2 - 2(1) = 2(1) + k,$$

so $k = -2$, and then $p = -k = 2$.

(f) We need

$$(1)^2 + k(1) + p \geqslant 1,$$

which rearranges to give $k + p \geqslant 0$, and we also need the positive gradient

$$2x + k > 0, \text{ for all } x > 1,$$

so $k > -2$. Notice that f is already increasing when $x < 1$.

(g) The function f is the sum of continuous functions ($x \mapsto \frac{\sin(x)}{x}$ is continuous when $x \neq 0$), so is also continuous on $\mathbb{R} \setminus \{0\}$.

$$\begin{aligned} &f(1) = 1 + \sin(1) - 3 < 0 \text{ and} \\ &f(3) = 27 + \frac{\sin(3)}{3} - 3 > 27 + \frac{-1}{3} - 3 > 17, \end{aligned}$$

so by the MVT (Mean-Value Theorem), there exists $\alpha \in (1,3) \subseteq \mathbb{R}$ satisfying $f(\alpha) = 17$. To find α where $f(\alpha) = 0$, consider the midpoint of $(1,3)$:

$$f(2) = 8 + \frac{\sin(2)}{2} - 3 = 5 - \frac{\sin(2)}{2} > 0.$$

Consider the midpoint of $(1,2)$:

$$f(1.5) > 0.$$

Consider the midpoint of $(1,1.5)$:

$$f(1.25) < 0.$$

The next midpoint is 1.375. Some students may be content with approximating $\alpha \approx 1.375$, but some may continuing the method further, increasing the accuracy of this approximation.

Continuing in this manner, we eventually conclude $\alpha \approx 1.31$.

(h) Showing that the limit has different values along any two lines is sufficient. We give three examples in this solution.

Approaching along the line $y = 3x^2$ gives

$$f(x, y) = \frac{1 - e^0}{2x^2 + 9x^4} \to 0.$$

Approaching along $x = 0$ and $y \to 0^+$ gives

$$f(x, y) = \frac{1 - e^{-y}}{y^4} \to \infty.$$

Approaching along $x = 0$ and $y \to 0^-$ gives

$$f(x, y) \to -\infty.$$

(i) M^{-1} exists if, and only if, $ad - bc \neq 0$, and is given by

$$M^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix};$$

$$\begin{aligned} MM^{-1} &= \frac{1}{ad - bc} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \\ &= \frac{1}{ad - bc} \begin{pmatrix} ad - bc & -ab + ab \\ cd - cd & -bc + ad \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I. \end{aligned}$$

(j)

$$A^{-1} = \frac{1}{22} \begin{pmatrix} 3 & -1 \\ -2 & 8 \end{pmatrix}$$

and B^{-1} does not exist since $3 \cdot 4 - 2 \cdot 6 = 12 - 12 = 0$.

(k)

$$\begin{aligned} (A + B)^T &= [a_{ij} + b_{ij}]^T \\ &= [a_{ji} + b_{ji}] \\ &= [a_{ji}] + [b_{ji}] \\ &= [a_{ij}]^T + [b_{ij}]^T \end{aligned}$$

$$= A^T + B^T$$

and

$$\begin{aligned}(AB)^T &= [a_{i1}b_{ij} + a_{i2}b_{2j}]^T \\ &= [a_{1i}b_{j1} + a_{2i}b_{j2}] \\ &= [b_{j1}a_{1i} + b_{j2}a_{2i}] \\ &= [b_{ji}][a_{ji}] \\ &= B^T A^T.\end{aligned}$$

With $C = B + I$, $(AC)^{-1}$ satisfies

$$\begin{aligned}(AC)(AC)^{-1} &= I \\ A(B+I)(AC)^{-1} &= I \\ \Rightarrow (AC)^{-1} &= (B+I)^{-1}A^{-1} \\ &= \begin{pmatrix} 4 & 2 \\ 6 & 5 \end{pmatrix}^{-1} \frac{1}{22} \begin{pmatrix} 3 & -1 \\ -2 & 8 \end{pmatrix} \\ &= \frac{1}{176} \begin{pmatrix} 19 & -21 \\ -26 & 38 \end{pmatrix}.\end{aligned}$$

(l)

$$\begin{aligned}AB &= \begin{pmatrix} 30 & 20 \\ 24 & 16 \end{pmatrix} \text{ and} \\ BA &= \begin{pmatrix} 28 & 9 \\ 56 & 18 \end{pmatrix}.\end{aligned}$$

(m)

$$\begin{aligned}\det(BA) = \det(B)\det(A) &= \frac{77}{3} \cdot (-39) \\ &= \frac{-3003}{3} = -1001.\end{aligned}$$

(n) Writing $A = [a_{ij}]$ and $B = [b_{ij}]$ ($i, j \in \{1, \dots, n\}$), the trace of $A + B$ is

$$\mathrm{tr}(A+B) = \sum_{k=1}^{n} (a_{kk} + b_{kk})$$

$$= \left(\sum_{k=1}^{n} a_{kk}\right) + \left(\sum_{k=1}^{n} b_{kk}\right)$$
$$= \mathrm{tr}(A) + \mathrm{tr}(B).$$

(o)

$$\det \begin{pmatrix} 8-\lambda & 1 \\ 2 & 3-\lambda \end{pmatrix} = 0,$$

so

$$\begin{aligned} 0 &= (8-\lambda)(3-\lambda) - 2 \\ &= 24 - 11\lambda + \lambda^2 - 2 \\ &= \lambda^2 - 11\lambda + 22 \\ \Rightarrow \lambda &= \frac{11}{2} \pm \frac{\sqrt{33}}{2}. \end{aligned}$$

(p) $A'A$ is an $n \times n$ matrix where each entry is 1, so $\mathrm{tr}(A'A) = n$.

AA' is a 1×1 matrix; $AA' = \begin{pmatrix} 1 \end{pmatrix}$, so $\mathrm{tr}(AA') = n$.

(q) We have

$$ABA' = \begin{pmatrix} 1 & \cdots & 1 \end{pmatrix} B \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix},$$

so $(1, \ldots, 1)B = [c_{1j}]$ is a $1 \times n$ matrix, with jth entry

$$c_{1j} = \sum_{i=1}^{n} 1 \cdot b_{ij} = \sum_{i=1}^{n} b_{ij},$$

so

$$AB = \left(\sum_{i=1}^{n} b_{i1}, \ldots, \sum_{i=1}^{n} b_{in}\right),$$

and therefore

$$ABA' = \left(\sum_{i=1}^{n} b_{i1}\right) + \cdots + \left(\sum_{i=1}^{n} b_{in}\right) = \sum_{i=1}^{n}\sum_{j=1}^{n} b_{ij}.$$

Therefore $b_{ij} = i + j$ gives

$$\begin{aligned}
ABA' &= \left(\sum_{i=1}^{n}(i+1)\right) + \cdots + \left(\sum_{i=1}^{n}(i+n)\right) \\
&= n + 2n + \cdots + n \cdot n + n\sum_{i=1}^{n} i \\
&= n\left(\sum_{i=1}^{n} i + \sum_{i=1}^{n} i\right) \\
&= 2n\sum_{i=1}^{n} i \\
&= 2n\frac{n(n+1)}{2} \\
&= n^2(n+1).
\end{aligned}$$

(r) With $b_{ij} = i + j^2$,

$$\begin{aligned}
ABA' &= \sum_{j=1}^{n}\sum_{i=1}^{n} b_{ij} \\
&= \sum_{j=1}^{n}\sum_{i=1}^{n} \left(i + j^2\right) \\
&= \sum_{j=1}^{n}\left(nj^2 + \sum_{i=1}^{n} i\right) \\
&= \sum_{j=1}^{n}\left(nj^2 + \frac{n(n+1)}{2}\right) \\
&= \frac{n^2(n+1)}{2} + n\sum_{j=1}^{n} j^2 \\
&= \frac{n^2(n+1)}{2} + \frac{n^2(n+1)(2n+1)}{6}.
\end{aligned}$$

(s) Consider $A = I$ and $B = -I$.