MATH7501 Assignment Solutions

These solutions produced by Mitchell Griggs.

1. Example 154 with $x^3 + x^2 + x$

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 + x^2 + 2xh + h^2 + x + h - (x^3 + x^2 + x)}{h} \\
&= \lim_{h\to 0} \frac{3x^2h + 3xh^2 + h^3 + 2xh + h^2 + h}{h} \\
&= \lim_{h\to 0} \left[3x^2 + 3xh + h^2 + 2x + 1\right] \\
&= 3x^2 + 2x + 1.
\end{aligned}$$

3. Example 156

$$\begin{aligned}
f(x) = \tan(x) &= \frac{\sin(x)}{\cos(x)} \\
\Rightarrow f'(x) &= \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)} = \sec^2(x).
\end{aligned}$$

3. Example 157 with $\sqrt[3]{\sin(x)}$

Set $u = \sin(x)$, so $y = u^{1/3}$, giving

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{1}{3}y^{-2/3}\cos(x) = \frac{1}{3}\frac{\cos(x)}{\sqrt[3]{\sin^2(x)}}.$$

4. Example 158 with $\log_2(x)$

With $y = \log_2(x)$, $x = 2^y = e^{y\log(2)}$, so

$$\frac{dy}{dx} = \frac{1}{\left(\frac{dx}{dy}\right)} = \frac{1}{\log(2)e^{y\log(2)}} = \frac{1}{\log(2)x}.$$

5. Example 159

Setting $y = \arcsin(x)$ gives $x = \sin(y)$, so

$$\frac{dy}{dx} = \frac{1}{\left(\frac{dx}{dy}\right)} = \frac{1}{\cos(y)} = \frac{1}{\sqrt{1-\sin^2(y)}} = \frac{1}{\sqrt{1-x^2}}.$$

6. Example 161

$$\begin{aligned}\lim_{x\to 0^+} x\log(x) &= \lim_{x\to 0^+} \frac{\log(x)}{(1/x)} \\ &= \lim_{x\to 0^+} \frac{(1/x)}{(-1/x^2)} = \lim_{x\to 0^+} (-x) = 0.\end{aligned}$$

7. Example 162

Define $f(x) = x^4 + 4x + 1$. We can observe

$$\begin{aligned} f(-2) = 9, \\ f(-1) = -2, \\ f(1) = 6, \end{aligned}$$

so by the IVT, there are definitely two solutions of $x^4 + 4x + 1 = 0$. To show that there are no more, we find the critical points:

$$f'(x) = 4x^3 + 4 = 0 \Leftrightarrow x^3 = -1 \Leftrightarrow x = -1,$$

so f only has one turning point, and therefore can't have more than two roots.

8. Example 166

The derivative f' is always positive; $f'(x) = 3x^2 + 1$, for all $x \in \mathbb{R}$, so f is strictly increasing on all of $\mathbb{R}$.

9. Example 168

$$\begin{aligned} f(x) &= 2x^3 + 3x^2 - 12x + 4 \\ \Rightarrow f'(x) &= 6x^2 + 6x - 12 \end{aligned}$$

$$\Rightarrow f''(x) = 12x + 6,$$

and the critical points satisfy

$$0 = f'(x) = 6x^2 + 6x - 12 \Rightarrow x \in \{-2, 1\}.$$

$f''(-2) < 0$ so f has a local minimum at -2.

$f''(1) > 0$ so f has a local maximum at 1.

10. Example 173

Due to gravity, $\frac{d^2y}{dt^2} = -g$ and $\frac{d^2x}{dt^2} = 0$, so

$$y = \frac{-1}{2}gt^2 + v_0 t \text{ and } x = u_0 t.$$

11. Example 176

$$f(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(x-1)^n,$$

with radius of convergence 1.

12. Example 178

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots,$$

with infinite radius of convergence.

13.

$$\exp(-x) = \sum_{n=0}^{\infty} \frac{(-1)^n \exp(-3)}{n!}(x-3)^n.$$

14.

(a) $f'(x) = 12x^3 - 48x^2 + 36x$

(b) $f'(x) = 3x^2 - 18x + 30$

(c) $f'(x) = 4x^3 - 6x^2 - 4x$

(d) $f'(x) = 5x^4$

(e) $f'(x) = 4x^3 + 3x^2$

(f) $f'(x) = 1 - \sin(x)$

(g) $f'(x) = 2\cos(2x) - 1$

(h) $f'(x) = \frac{1}{x} - 3x^2$, for $x > 0$.

(i) $f'(x) = 2^x \log(2) - 1$

(j) $y = \frac{1}{2}x^4 - \frac{1}{2}x^2 + c$

(k) $y = -\cos(x) - 6x^2 + c$

(l) $y = \frac{x^3}{3} + 3x + c$ and $y(0) = -2$ give $y = \frac{x^3}{3} + 3x - 2$.

(m) $y = \frac{x^2}{2} + 2\sin(x) + c$ and $y(0) = 3$ give $y = \frac{x^2}{2} + 2\sin(x) + 3$.

Derivative tests are applied to find and classify the critical points of the functions in the following problems.

(n) 0 point of inflection

1 local maximum

3 local and global minimum

-1 global maximum

(o) -4 global minimum

6 global maximum

(p) $-1/2$ local minimum

0 local maximum

2 local and global minimum

3 global maximum

(q) 0 turning point

-2 global minimum

2 global maximum

(r) $-3/4$ local and global minimum

0 point of inflection

3 global maximum

(s) $-3\pi/2$ point of inflection

$\pi/2$ point of inflection

-2π global minimum

2π global maximum

(t) $-11\pi/6$ local and global maximum

$-7\pi/6$ local minimum

$-5\pi/6$ local maximum

$-\pi/6$ local minimum

$\pi/6$ local maximum

$5\pi/6$ local minimum

$7\pi/6$ local maximum

$11\pi/6$ local and global minimum

(u) $\frac{1}{\sqrt[3]{3}}$ local and global maximum

No minimum exists; $\lim_{x\to 0^+}(\log(x) - x^3) = -\infty$.

(v) $-\log_2(\log(2))$ local and global minimum

5π global maximum