MATH7501 Assignment Solutions

These solutions produced by Mitchell Griggs.

1. Example 192

The substitution $u = x^4 + 2$ yields

$$\int x^3 \cos\left(x^4+2\right)\, dx = \frac{1}{4}\sin\left(x^4+2\right) + c.$$

2. Example 193

The substitution $u = 5x - 3$ yields

$$\int_1^2 \frac{dx}{(5x-3)^2} = \frac{1}{14}.$$

3. Example 194

The substitution $x = a\sin\theta$ yields

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin(x/a) + c.$$

4. Example 198

The substitutions $u = x$ and $v = e^x$ allow for us to apply integration by parts, to conclude

$$\int xe^x\, dx = xe^x - e^x + c.$$

5.

$$\int_0^\infty x^2 e^{-x}\, dx = \left[-x^2e^{-x} - 2xe^{-x} - 2e^{-x}\right]_0^\infty = 2,$$

by integrating by parts, twice.

6. Example 199

$$\int \frac{x+2}{x^2+x}\, dx = \int \frac{x+2}{x(x+1)}\, dx$$

$$
\begin{aligned}
&= \int \frac{2}{x}\,dx - \int \frac{1}{x+1}\,dx \\
&= \log|x| - \log|x+1| + c.
\end{aligned}
$$

7. Example 200

$$
\begin{aligned}
\int \frac{dx}{x^2 - a^2} &= \frac{1}{2a} \int \left(\frac{1}{x-a} - \frac{1}{x+a} \right) dx \\
&= \frac{1}{2a} \left(\log|x-a| - \log|x+a| \right) + c.
\end{aligned}
$$

8.

Rectangular prism:

$$
\int_0^h \int_0^l \int_0^w dx\,dy\,dz = lwh.
$$

Cylinder:

$$
\int_0^h \int_0^{2\pi} \int_0^R r\,dr\,d\theta\,dz = 2\pi Rh.
$$

Sphere:

$$
\int_0^\pi \int_0^{2\pi} \int_0^R r^2 \sin\phi\,dr\,d\theta\,d\phi = \frac{4\pi R^3}{3}.
$$

There are many other examples.

9. Example 201

$$
\int_0^2 \int_1^3 x^2 y\,dy\,dx = 32/3.
$$

10. Example 202

$$
\int_1^3 \int_0^2 x^2 y\,dx\,dy = 32/2.
$$

11. Example 204

$$\int_0^\infty e^{-x}\,dx = \lim_{a\to\infty}\int_0^a e^{-x}\,dx = \lim_{a\to\infty}\left[-e^{-x}\right]_0^a = 1.$$

12.

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dx\,dy = \int_0^{2\pi}\int_0^{\infty} re^{-r^2}\,dr\,d\theta = \int_0^{2\pi}\frac{1}{2}\,d\theta = \pi.$$

13.

$$\left\{(x,y)\in\mathbb{R}^2 : |(x,y)| = r\right\},$$

or this set may be written in other ways, such as

$$\left\{(x,y)\in\mathbb{R}^2 : x^2+y^2 = r^2\right\}.$$

14.

We want to show that for any n-dimensional vectors $\mathbf{u},\mathbf{v}\in\mathbb{R}^n$, the inequality

$$|\mathbf{u}\cdot\mathbf{v}| \leqslant |\mathbf{u}|\,|\mathbf{v}|$$

holds.

To show this, we apply a trick and instead begin with the inequality $\mathbf{w}\cdot\mathbf{w}\geqslant 0$, which holds for every $\mathbf{w}\in\mathbb{R}^n$. In particular, if we let $x\in\mathbb{R}$ and $\mathbf{w}=\mathbf{u}+x\mathbf{v}$, then we have

$$\begin{aligned} 0 \leqslant (\mathbf{u}+x\mathbf{v})\cdot(\mathbf{u}+x\mathbf{v}) &= \mathbf{u}\cdot\mathbf{u} + 2x\mathbf{u}\cdot\mathbf{v} + x^2\mathbf{v}\cdot\mathbf{v} \\ &= (\mathbf{v}\cdot\mathbf{v})x^2 + (2\mathbf{u}\cdot\mathbf{v})x + (\mathbf{u}\cdot\mathbf{u}). \end{aligned}$$

This quadratic's discriminant (b^2-4ac) must not be positive (if it were then the function f, given by

$$f(x) = (\mathbf{v}\cdot\mathbf{v})x^2 + (2\mathbf{u}\cdot\mathbf{v})x + (\mathbf{u}\cdot\mathbf{u}),$$

would have two real zeroes, meaning it would be negative somewhere), so

$$0 \geqslant b^2-4ac = (2\mathbf{u}\cdot\mathbf{v})^2 - 4(\mathbf{v}\cdot\mathbf{v})(\mathbf{u}\cdot\mathbf{u}),$$

meaning

$$(\mathbf{v}\cdot\mathbf{v})(\mathbf{u}\cdot\mathbf{u}) \geqslant (\mathbf{u}\cdot\mathbf{v})^2,$$

so

$$(\mathbf{u}\cdot\mathbf{v})^2 \leqslant |\mathbf{u}|^2\,|\mathbf{v}|^2\,,$$

giving

$$|\mathbf{u}\cdot\mathbf{v}| \leqslant |\mathbf{u}|\,|\mathbf{v}|\,.$$

15. Example 208

The direction is

$$\mathbf{w}_1 = \left(\frac{\mathbf{w}\cdot\mathbf{v}}{|\mathbf{v}|^2}\right)\mathbf{v} = \begin{pmatrix}4\\4\end{pmatrix},$$

so the projection of **w** onto **v** is

$$\mathbf{w}_2 = \mathbf{w} - \mathbf{w}_1 = \begin{pmatrix}2\\-2\end{pmatrix}.$$

16. (98)

(a)

$$\int \left(x^3 + 3x\right)\,dx = \frac{x^4}{4} + \frac{3x^2}{2} + c.$$

(b)

$$\int x^3\cos\left(x^4+2\right)\,dx = \frac{1}{4}\sin\left(x^4+2\right) + c.$$

(c)

$$\int_1^2 \frac{dx}{(5x-3)^2} = \frac{1}{84},$$

by the substitution $u = 5x - 3$.

(d)

$$\int \frac{dx}{\sqrt{9-x^2}} = \arcsin(x/3) + c.$$

(e)

$$\int \cos(x)\sin(x)\,dx = \frac{-\cos^2(x)}{2} + c.$$

(f)

$$\int (3x+1)\left(3x^2+2x\right)^3\, dx = \frac{\left(3x^2+2x\right)^4}{8} + c,$$

by the substitution $u = 3x^2 + 2x$.

(g)

$$\int x^3 \log(x)\, dx = \frac{x^4}{16}(4\log(x) - 1) + c.$$

(h)

$$\int xe^x\, dx = (x-1)e^x + c.$$

(i)

$$\int \frac{x+2}{x^2+x}\, dx = 2\log|x| - \log|x+1| + c.$$

(j)

$$\int \frac{dx}{x^2-16} = \int \frac{dx}{(x+4)(x-4)} = \frac{\log|4-x| - \log|x+4|}{8} + c.$$