— PRACTICE EXAM —

Instructions

The exam consists of 5 questions, 1– 5. Each question has three items, a – c.

Within each question:

Item (a) carries a weight of 10 marks.

Item (b) carries a weight of 6 marks.

Item (c) carries a weight of 4 marks.

Answer ALL questions in the spaces provided.

If more space is required, use the back of the PREVIOUS page.

Show all your working and include sketches where appropriate.

Please do not write on the course reader booklets supplied to you.

Question 1:

Consider the set, $\mathcal{W}$ as the set of all valid words in the English language made up only of the letters a–z. That is, ignore punctuation marks and other symbols. Treat a word as a tuple of letters - for example the word "giant" is the tuple, (g, i, a, n, t). As examples observe $(d, o, g) \in \mathcal{W}$ and $(s, t, o, n, e) \in \mathcal{W}$ but $(d, d, d, d, j, f, a) \notin \mathcal{W}$.

Let $\mathcal{C} = \{a, b, c, d, \dots, x, y, z\}$ be the set of lower case English letters. And denote $\mathcal{W}_n := \mathcal{W} \cap \mathcal{C}^n$ for $n = 1, 2, 3, \dots$.

(1a) Use your knowledge of the English language to evaluate $|\mathcal{W}_1|$ (the cardinality of that set).

(1b) Consider now the sequence, $a_1, a_2, a_3, \ldots$ with $a_n = \big|\mathcal{W}_n\big|$. Further define the sequence $b_1, b_2, b_3, \ldots$ via,

$$b_n := \frac{a_n}{|\mathcal{W}|}.$$

Explain the meaning of the expression,

$$\sum_{n=1}^{\infty} n\, b_n.$$

(1c) Let R be a binary relation on $\mathcal{W}$ where xRy if and only if the first letter of x is the first letter of y. Consider now,

$$U = \{A \subset [(g,i,a,n,t)]_R \ : \ |A| = 3\}.$$

Write down an element of U.

Question 2:

Let $\alpha_0, \alpha_1, \beta_0$ and β_1 be specified parameters. Consider the sequence $\{x_n\}_{n=0}^{\infty}$ where x_0 is given and the other elements follow the recursion

$$x_{n+1} = \begin{cases} \alpha_0 x_n + \beta_0 & \text{if } \ x_n \equiv 0 \pmod 2, \\ \alpha_1 x_n + \beta_1 & \text{if } \ x_n \equiv 1 \pmod 2. \end{cases}$$

(2a) Assume that $\alpha_0 = \alpha_1 := \alpha$ and $\beta_0 = \beta_1 = 0$. For what values of α does the limit of $\{x_n\}$ exist?

(2b) Assume that $\alpha_0 = \frac{1}{2}$, $\beta_0 = 0$, $\alpha_1 = 3$ and $\beta_1 = 1$. Let $x_0 = 5$ determine x_n for all n.

(2c) Consider now again the parameters as specified in (2b) above. In that case, computers have been used to verify that when $a_0 \in \{1, 2, \ldots, N\}$ it holds that $a_n = 1$ for some n. This has been tried out for very large N. Consider the logical statement,

$$P_k = \text{“For } a_0 = k \text{ it holds that } a_n \neq 1 \text{ for all } n\text{”}.$$

It is conjectured that for all k, $\neg P_k$ and a proof or counter example is still not known.

As a start, find an arbitrarily large k such that $\neg P_k$ holds.

Question 3:

The $\operatorname{sinc}(x)$ function defined as

$$\operatorname{sinc}(x) = \frac{\sin(x)}{x},$$

is extremely important in signal processing.

(3a) Evaluate the limit

$$\lim_{x \to 0} \operatorname{sinc}(x).$$

(3b) It is known that,

$$\operatorname{sinc}(x) = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{(2n+1)!}.$$

Derive this result.

(3c) It holds that,

$$\int_0^1 \operatorname{sinc}(x)\,dx \approx \frac{1703}{1800}.$$

Use the series from (3b) to show this.

Question 4:

Consider the function $f(x) = Ke^{-2x}\mathbf{1}\{x \geq 0\}$ with K some constant. Here $\mathbf{1}\{x \geq 0\}$ is 1 for $x \geq 0$ and otherwise is 0.

(4a) Say you wish to have $\int_{-\infty}^{\infty} f(x)\,dx = 1$. Find K that achieves this.

(4b) Take now $K = 1$ and let $g(x) = f(x) + f(x-1)$. Find the global maximum of $g(x)$ over $x \in \mathbb{R}$.

(4c) Use $g(\cdot)$ from above. Compute $\int_{-\infty}^{\infty} x^2 g(x)\, dx$.

Question 5:

Consider a function $F : \mathbb{R} \to \mathbb{R}^{2\times 2}$ defined as follows,

$$F(x) = \begin{bmatrix} \cos(x) & -\sin(x) \\ \sin(x) & \cos(x) \end{bmatrix}.$$

(5a) Show that $F(x) * F(-x)$ is the identity matrix (where $*$ is matrix multiplication).

(5b) Is $F(x)$ singular for any value of x? If so for what value? If not, prove your result.

(5c) Consider now the function $u : [0, \pi] \to \mathbb{R}$ defined by,

$$u(x) = [0 \;\; 1] * F(x) * \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Find $x \in [0, \pi]$ that maximises $u(x)$.

END OF EXAMINATION