1. In this exercise, consider functions of the form $f : \mathbb{R}^2 \to \mathbb{R}$.

   (i) Take $f_a(x, y) = a \cdot [x\ y]'$ for a vector $a \in \mathbb{R}^2$, where $\cdot$ is the dot product. For what vectors a is f_a an onto function? For what values is it a one-to-one function? Are there any vectors a for which it is an onto function and not a one-to one function? In case it isn't an onto function, what is the range?

   Take now $f_A(x, y) = [x\ y]\, A\, [x\ y]'$ for some matrix A.

   (ii) Consider $A = 2I$. Plot a contour plot and a 3D plot of f_A. Include the origin, $(x, y) = (0, 0)$ in your plot. Is the function an onto function?

   (iii) Find a matrix A for which f_A is an onto function.

   (iv) Is there a matrix A for which f_A is a one-to-one function? If yes, present one such A. Otherwise, try to prove or argue why not.

   (v) Try to describe the class of matrices A, for which f_A posseses a global minimum or a global maximum. If you are not able to do that, present 3 examples that don't posses a global minimum or maximum and 3 examples that do posses such an extremum point.

2. Use a truth table to show that the following logical statement is a tautology:

$$(P \to Q) \vee (Q \to P) \vee (\neg Q).$$

3. Consider sequences of numbers of the form $x_n = n^\alpha$ for $n = 1, 2, 3, \dots$ with $\alpha \in \mathbb{R}$.

   (i) For what values of α does $\lim_{n\to\infty} x_n = 0$?

   (ii) Set now $S = \sum_{k=1}^{\infty} x_k$. For what values of α does the series, S, converge?

   (iii) Consider $\alpha = -2$. Use Mathematica to analytically evaluate S. Use this result to suggest an algorithm for numerically approximating the constant π and implement it in Mathematica.

   (iv) Consider $\alpha = -1$. In this case S is called the harmonic series. It holds that,

$$\sum_{k=1}^{n} x_k = \log(n) + \gamma + e_n,$$

   where γ is Euler's gamma constant and e_n is an $o(1)$ sequence. Numerically approximate γ and plot the sequence e_n.

4. Consider the series $\sum_{k=0}^{\infty} \frac{3^k}{k!}$, use the ratio test to show that it converges.

5. Let $\mathbf{1}_n$ be a column vector of 1's and b_n be the vector $[1\ 2\ 3\ \dots\ n]'$.

$$x_n = \frac{1}{n^\alpha} \mathbf{1}_n' \left(b_n\ b_n' \right) \mathbf{1}_n$$

   Is there a α for $\lim_{n\to\infty} x_n$ if the limit exists? If so, for which α does the limit exist?

6. Use the formal (ϵ, N), definition of the limit of a sequence to show that if $\lim_{n\to\infty} x_n = L_1$ and $\lim_{n\to\infty} y_n = L_2$ then,

$$\lim_{n\to\infty} x_n + y_n = L_1 + L_2.$$