1. Consider the following probability density functions. Use a brute-force Riemann sum to show that the area under the curve is 1.

   (i) $f(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$ for all real x.

   (ii) $f(x) = e^{-x}$ for $x \geq 0$.

   (iii) $f(x) = \frac{1}{\sqrt{2\pi}}x^{-1/2}e^{-x/2}$ for $x \geq 0$.

   (iv) $f(x) = \frac{1}{\pi(1+x^2)}$ for all real x.

   (v) $f(x) = xe^{-x^2/2}$ for $x \geq 0$.

   (vi) $f(x) = 30x(1-x)^4$ for $x \in [0,1]$.

2. For each of the probability density functions of the previous question try to show that area under the curve is 1 analytically. Note that (i), (iii) and (vi) require more complicated methods (than substitution or integration by parts) . Try and look these up and explain.

3. For each of the probability density functions above, compute the mean (also known as expected value) by computing,

$$\int_{x \in S} x f(x)\, dx,$$

   where S is the set of values on which the function is defined. If you are able to compute it analytically then do it numerically. For (iv), the mean is not defined. Explain why.

4. The median of the probability density function is defined as the value γ such that,

$$\int_m^{\gamma} f(x)\, dx = \frac{1}{2},$$

   where m is the minimal value on which the density is defined. Try and calculate the median for each of the distributions above. For certain cases, reason based on symmetry.

5. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by, $f(x) = (x-a)^T A(x-a)$ where

$$A = \begin{bmatrix} 4 & -2 \\ -2 & 4 \end{bmatrix},$$

   and $a = [5 \;\; 4]^T$. Implement a gradient descent algorithm for finding the minimal value of $f(x)$ starting at an initial value of $x_0 = [10 \;\; 10]^T$. Experiment with different learning rates to see when the algorithm converges. Present your results in a neat graphical manner.

6. Consider the sinc() function,

$$\text{sinc}(x) = \frac{\sin(x)}{x}.$$

   (i) Derive the Taylor series for it about $x = 0$.

   (ii) Say you wish to approximate it via a finite polynomial over the range $[-3\pi, 3\pi]$ with an error of no more than 10^{-3}. What is the minimal number of terms needed?

   (iii) Search for at least two alternative derivations of the following and explain the steps:

$$\int_{-\infty}^{\infty} \text{sinc}(x)\, dx = \pi.$$

7. Consider a time signal $u(t) = \cos(t)$. You are smoothing it with the running average of length T:

$$v(t) = \frac{1}{T}\int_0^T u(t-\tau)\, d\tau.$$

   Compute $v(t)$ and reason about the amplitude of $v(t)$ as T varies. Plot some examples.