MATHEMATICA BASED EXERCISES FOR MATH7501

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(1) Let $n = 155$. Use the floor function $\lfloor \cdot \rfloor$, "Floor" to express $n = 3q + r$. Remember to use "ClearAll" at the end.

(2) A symbolic exercise: Use *Mathematica* to help formulate a proof that if $a - b = rm$, $b - c = rm$, then $a - c = tm$ for $r, s, t \in \mathbb{Z}$. You can rearrange the expressions so that the right hand side of each equation is zero, then add them, and finally use the function "Factor". This will give a formula for t.

(3) The following expressions

```
"||", "&&", "Not"
```

denote OR, AND, and NOT respectively, while "True" and "False" mean true and false. Determine the truth value of $p \wedge (q \vee \sim r)$ when p is false, q is false, and r is false.

(4) Use "Mod" to determine whether $9 \mid 129$. Is 129 a multiple of 3?

(5) The following

```
Sum[j, {j, 1, 8}]
```

calculates $\sum_{j=1}^{8} j$. Use a similar method to calculate $\sum_{j=1}^{12} j^2$.

(6) Use "Binomial" to calculate the binomial coefficient $\binom{5}{3}$ and then use the function "Factorial" to check this. Use Part (e) to help you calculate $\sum_{r=1}^{n} \left(\binom{n}{r-1} + \binom{n}{r} \right)$ for $n = 11$.

(7) Define a function called 'carl', using local variables via Module[], which inputs two numbers x, y and returns $(x + y)^2$.

```
carl[x_, y_] := Module[{p, q}, p = 7; (x + y)^2];
carl[3, 5]
```

What are p and q doing? Try changing the function.

(8) Let $A = \{-1, 7\}$, $B = \{1, 8, 7, -1, 4\}$, and $Q = \{1, -1\}$, $R = \{5, 7\}$. Determine the cardinality of B, compute whether or not $8 \in B$, whether $A \subseteq B$, $B \cap Q$, $B \cup R$, $B \setminus Q$, and the join of B and Q using the following:

```
A = {-1, 7}; B = {1, 8, 7, -1, 4};
Q = {1, -1}; R = {5, 7};
Length[B]
MemberQ[B, 8]
```

```
Intersection[B, Q]
Union[B, Q]
SubsetQ[B, Q]
Complement[B, Q]
Join[B, Q]
ClearAll[A, B, Q, R]
```

Construct and then display the matrix 2×3 M obtained from $B \cup R$ by using the function "Partition".

(9) We might construct the truth table for $(p \vee q) \wedge (\sim p)$ as a matrix as follows:

```
Z = {p, q};
m = Length[Z];
n = 2^m
A = Table[PadRight[IntegerDigits[g, 2], m], {g, 0, n - 1}]
B = ReplaceAll[A, {0 -> False, 1 -> True}]
carol[B_, j_] :=
  Module[{p, q}, p = B[[j, 1]]; q = B[[j, 2]]; (p || q) && Not[p]];
L = Table[carol[B, j], {j, 1, 4}]
M = Table[{B[[k]], L[[k]]}, {k, 1, 4}]
MatrixForm[M]
ClearAll[A, B, L, M, Z, n]
```

Try this and then do the same for $((p \vee q) \longrightarrow r) \longleftrightarrow (p \longrightarrow r) \wedge (q \longrightarrow r)$. What can you conclude? Can you use this method to prove De Morgan's laws?

(10) Try out the following and describe what it does:

```
m = {{1, 2}, {3, 4}};
s = {{3, 3, 5}, {0, 0, 0}};
t = {{18, -3}, {8, 9}};
Q = {{m, m, t}, {m, m, m}};
MatrixForm[Q]
S = ArrayFlatten[Q];
MatrixForm[S]
```

How would you turn S into a list/set by the process known as the vectorisation of a matrix?

(11) Let A and B be the sets in the previous question. Using "Table" and $A[[j]]$, $B[[k]]$, construct the Cartesian product $M = A \times B$ and then display the result as a matrix. If you know or use LaTeX, copy the result via "copy as latex" and paste it somewhere.

(12) Use ContourPlot to plot $5x^2y^2 + y^3 = 4x$ in the domain $[-3,3]^2$. Use the same method with three circles of radius 1.5 centered at $(0,0)$, $(1,1)$, and $(-1,1)$ to plot a Venn diagram. Use the function "Text" to fill in your diagram with the elements of the sets $A = \{1,2,3\}$, $B = \{5,2,3\}$, $G = \{9,3,8\}$. You may need to adjust the circles to make a useful diagram.

(13) There is a well know diagram which illustrates that the cardinality of the rational numbers is equal to that of the integers. Construct a function $f(x)$ which takes a positive integer x and produces the rational number corresponding to Fig. 1.

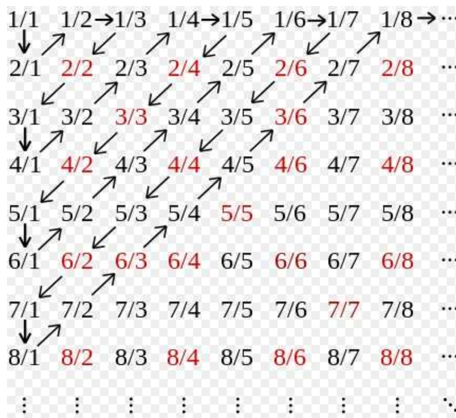

FIGURE 1. Rational numbers are countable

(14) Verify computationally that for all odd primes $3 < p < 100000$, $p \equiv \pm 1 \pmod 6$ using:

```
S = Table[
   p = Prime[j];
   q = Mod[p, 6];
   If[(q == 1) || (q == 5), 1, 0], {j, 1, 78498}];
ClearAll[p, q]
Position[S, 0]
```

Discuss what is occurring in each line of this code, identify logical symbols, and how this achieves the task.

(15) The polynomial $f(x) = x^2 + x + 41$ is famous for being simple looking while having many consecutive values of $x \in \mathbb{Z}$ such that $f(x)$ is prime. Disprove the following claim "For all integers x, $f(x)$ is prime" by finding a counter-example. In comment form: "(* A comment *)", explain the logic showing how this disproves the claim.

(16) Using only the following built-in functions: Table, If, Length, Partition, RotateRight, DeleteCases, PrimeQ, form a set S of the first 100 squares plus 1, remove those elements of S which are prime numbers, then partition the resulting set into triples, and finally rotate the set 5 places to the right.

(17) Using the functions: IntegerDigits, Reverse, FromDigits, give the binary representation of 221, reverse sort the list, and then produce the base 10 number from the resulting binary number.

(18) Use the function PowerMod to show that the number $n = 5519$ is probably prime by calculating $2^n \pmod n$, $3^n \pmod n$, and $5^n \pmod n$. Also check this the slow way.

(19) Make an image:

```
Image[ {{0, 1, 0}, {1, 0, 0}, {0, 1, 0}}]
```

(20) Copy the file "cat.jpg" from BlackBoard and put it in the MyDocuments folder. Now import the image, obtain the raw data from it and output the first two elements of the set (first two rows of the array):

```
L = Import["cat.jpg"];
M = ImageData[L];
Take[M, 2]
```

Now construct a set P by transposing the set M and output the graphic image of P.

```
P = Take[L, 300];
Q = Transpose[P];
Image[Q, "Real", ColorSpace -> "RGB", Interleaving -> True]
```

Use

```
Length[Partition[Flatten[M], 3]]
```

to determine how many pixels the image L contains.

(21) Try the following and then explain what is occurring:

```
m = Length[P[[1]]]
R = Take[P, m];
S = Transpose[R];
Image[R + S, "Real", ColorSpace -> "RGB", Interleaving -> True]
```

Use 'Random' to add noise to the image.

(22) Try to compute $\int_0^{20} e^{-x^2}\,dx$ in several ways:

```
Integrate[Exp[-x^2], x]
Integrate[Exp[-x^2], {x, 0, 20}]
NIntegrate[Exp[-x^2], {x, 0, 20}]
```

and using the trapezoidal rule and "Sum".

(23) Show that the following are logically equivalent. $p \Longrightarrow q \vee r$, $p \wedge \sim q \Longrightarrow r$, and $p \wedge \sim r \Longrightarrow q$ by constructing a truth table similar to the way we can construct a truth table for $(p \vee q) \wedge (\sim p)$: (Hint: Modify the function "carol")

```
Z = {p, q};
m = Length[Z];
n = 2^m
A = Table[PadRight[IntegerDigits[g, 2], m], {g, 0, n - 1}]
B = ReplaceAll[A, {0 -> False, 1 -> True}]
carol[B_, j_] :=
  Module[{p, q}, p = B[[j, 1]]; q = B[[j, 2]]; (p || q) && Not[p]];
L = Table[carol[B, j], {j, 1, 4}]
```

```
M = Table[{B[[k]], L[[k]]}, {k, 1, 4}]
MatrixForm[M]
ClearAll[A, B, L, M, Z, n]
```

(24) Use "NSolve" to find the roots of $x^2 - 39$.

(25) Calculate $\sqrt{3}$ to 200 decimal places. Does the decimal expansion of $\sqrt{3}$ repeat? What can you conclude about the rationality of $\sqrt{3}$?

(26) We can construct a list of the values of $n^2 + 5$ for $n = 1, 2, \ldots, 25$ as follows:

```
Table[n^2 + 5 , {n, 1, 25}]
```

Use this and the function "Mod" to determine whether any of these are divisible by 4.

(27) By an exhaustive search, determine whether $7m + 4$ is divisible by 7 for $m \in \mathbb{Z}$ and $0 < m < 10000$.

(28) By an exhaustive search, where $0 < a, b < 50$, determine whether it is likely that the following holds: If $a, b, c \in \mathbb{Z}$ and $a^2 + b^2 = c^2$, then at least one of a and b is even. Hint: Use the function "EvenQ[]" and tabulate $\sqrt{a^2 + b^2}$ using a, 1, 50, b, 1, 50.

(29) Find the prime factorisation of $n = 4804255585375$ using "FactorInteger[n]".

(30) Test the conjecture: If product of two positive real numbers is greater than 100, then at least one of the numbers is greater than 10.

(31) Test the conjecture: $\sum_{j=1}^{n} j^3 = \frac{1}{4} n^2 (n+1)^2$ using:

```
Table[{Sum[j^3, {j, 1, n}], (1/4)*n^2 (n + 1)^2}, {n, 1, 100}]
```

(32) Plot the function $f(n) = \frac{1}{4} n^2 (n+1)^2$ in the interval $[1, 10]$ together with the points $(n, f(x))$. Hint:

```
f = (1/4) n^2 (n + 1)^2;
A = Plot[f, {n, 1, 10}];
S = Table[{n, f}, {n, 1, 10}];
B = ListPlot[S, PlotStyle -> PointSize[0.02]];
Show[A, B]
```

(33) Test the conjecture for $n < 10^4$: For integers $n \geq 2$,

$$\prod_{i=2}^{n} \left(1 - \frac{1}{i^2}\right) = \frac{n+1}{2n}.$$

(34) Test the conjecture for $n < 10^4$: For all integers $n \geq 2$,

$$2n + 1 < n^3.$$

(35) We can find a formula for $\sum_{j=1}^{n} j^4$ as follows:

```
F = a n^5 + b n^4 + c n^3 + d n^2 + f n + g
S = Table[F - Sum[j^4, {j, 1, n}], {n, 1, 6}]
```

```
Solve[S == Table[0, {n, 1, 6}]]
```

Find a formula for $\sum_{j=1}^{n} j^9$ and express it in factorised form using "Factor".

(36) Optional challenge: Let $f_m(n) = \sum_{j=1}^{n} j^m$. Find the rational numbers $f_m(0)$ for $m = 1, 2, 3, \ldots, 11$.

(37) Try the following in *Mathematica*:

```
Graph[{1 -> 2, 2 -> 3, 3 -> 1, 3 -> 2, 1 -> 1},
       VertexLabels -> "Name"]
```

(38) Input the following in *Mathematica*: $A = \{1, 2, 3, 4\}$ and

$$R = \{\{1,1\}, \{1,2\}, \{2,1\}, \{2,2\}, \{3,4\}, \{4,1\}, \{4,4\}\}.$$

Plot a labeled directed graph from R. Use "Table" and indexing of R to plot a directed graph of R. For example, using

```
Table[R[[j,1]] - > R[[j,2]], {j, 1, Length[R]}]
```

(39) Include additional relations in R, forming R' to make R' an equivalence relation.

(40) Find the equivalence classes of R', in (3).

(41) C is the relation on real numbers: xCy iff $x^2 + y^2 = 1$. Use "ContourPlot" to plot the circle.

(42) F is the relation on $\mathbb{Z}$: mFn iff $5 \mid (m - n)$. Use "Mod" to find 10 distinct members of the equivalence class containing -7.

(43) Let $f(x) = \log(x)$ and $g(x) = \cosh(x)$. Plot the functions $f(g(x))$ and $g(f(x))$.

(44) Consider the function $f : \mathbb{R} \longrightarrow \mathbb{R}$ given by $f(x) = x^3 + 3x - 5$. Find $f^{-1}(x)$ and plot f^{-1} together with f on $[-3, 3]$.

(45) Is the function $f : \mathbb{R} \longrightarrow \mathbb{R}$ given by $f(x) = \frac{1}{x^2+2}$ one to one? onto? Does f have an inverse? If the domain of f is restricted to the set of positive real numbers $\mathbb{R}^+$, would f have an inverse? If so find f^{-1}.

(46) Sketch the plane $-3x + 2y - z = 6$.

(47) Use "ListPointPlot3D" to plot the points $(1, 9, -4)$, $(6, -2, 4)$, $(1, 0, 9)$ and connect these points by lines:

```
S = {{1, 9, -4}, {6, -2, 4}, {1, 0, 9}};
A = ListPointPlot3D[S, PlotStyle -> PointSize[0.02]];
B = Graphics3D[{Line[{S[[1]], S[[2]]}], Line[{S[[1]], S[[3]]}],
    Line[{S[[2]], S[[3]]}]}];
Show[A, B]
```

Find rational numbers a, b, c such that $ax + by + cz = 1$ using "Solve" and the following list of equations:

```
T = Table[a S[[j,1]] + b S[[j,2]] + c S[[j,3]] == 1 ,
    {j, 1, Length[S]}]
```

(48) Plot the surface $z = (x^2 - 3y^2)^{-1}$ using "ParametricPlot3D".

(49) Plot the surface $z = x^2 - 3y^2$ using "Plot3D"

(50) Plot the surface $z^3 = x^2 - 3y^2$ using "ContourPlot3D" in the region $-2 \le x, y, z \le 2$.

(51) Optional Challenge: Use the functions "Random[]", "Table", and "ListPointPlot3D" to generate and plot data points near the surface $z^3 = x^2 - 3y^2$.

(52) Plot the following functions near $x = 1$ and $x = 2$ respectively

$$f(x) = \frac{x^{1000} - 1}{x - 1} \qquad\qquad g(x) = \frac{\sqrt{(6-x)} - 2}{\sqrt{(3-x)} - 1}.$$

(53) Determine the first 20 terms of the following sequences and use "ListPlot" to plot the terms.
(a) $d_n = 4 - d_{n-1},\ d_1 = 1$
(b) $e_n = 4 - e_{n-1},\ \ e_1 = 2$

(54) Compute the first 10 terms of the following sequences using "N[]" where appropriate :

(a) $a_n = \ln(n) - \ln(3n+2)$	(b) $a_n = \left(1 + \frac{2}{n}\right)^n$	(c) $a_n = \frac{2n}{5n-3}$
(d) $a_n = \frac{n^2 - n + 7}{2n^3 + n^2}$	(e) $1 + \left(\frac{9}{10}\right)^n$	(f) $1 + (-1)^n$
(g) $a_n = \frac{\sin^2 n}{\sqrt{n}}$	(h) $n \sin(\pi n)$	(i) $a_n = \pi^{-(\sin n)/n}$
(j) $\frac{\tan^{-1}(n)}{n}$		

(55) Use "ListPlot" to plot the terms of the following sequences and try to find functions which bound the terms:
(a) $a_n = \frac{n!}{n^n}, \quad n = 1, 2, \ldots$
(b) $b_n = \left(\frac{3^n + 1}{n}\right)^{1/n}$ $n = 1, 2, \ldots$ (Recall the standard limit $\lim_{n\to\infty} n^{1/n} = 1$)

(56) For several values of p, compute $\sum_{n=1}^{20} \frac{(-1)^n}{n+p}$.

(57) Calculate and plot the terms of the series $\sum_{n=1}^{m} (-1)^n \frac{n^3}{3^n}$, for $m = 1, 2, \ldots 10$.

(58) Calculate and plot the terms of the following series for $m = 1, 2, \ldots 10$:
(a) $\sum_{n=1}^{\infty} \frac{\sin(4n)}{4^n}$,
(b) $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln(n)}$,

(c) $\sum_{n=2}^{\infty} \frac{(2n)!}{(n!)^2}$.

(59) Plot the function $f(x) = \frac{x^3+2x^2-1}{5-3x}$ near $x = -2$.

(60) Plot the function $f(x) = x^4 \cos\left(\frac{2}{x}\right)$ near $x = 0$.

(61) Plot the following functions:
(a) $f(x,y) = \frac{xy^2}{x^2+y^4}$.
(b) $f(x,y) = \frac{xy\sin(x+1)}{x^2+2y^2}$.
(c) $f(x,y) = \frac{x^2+2xy+2y^2}{3x^2+2y^2}$.

(62) Try the following code:

```
f = 1 - x^2 - y^2;
A = Plot3D[f, {x, -2, 2}, {y, -2, 2},
   BoxRatios -> {1, 1, 1}];
B = ParametricPlot3D[{t, -1, -t^2}, {t, -2.5, 2.0}, PlotRange -> All,
   PlotStyle -> Tube[0.1]];
G = Show[A, B]
```

By modifying the functions involved, plot the surface $z = \frac{xy^2}{x^2+y^4}$ together with two paths which pass through the point $(x,y,z) = (0,0,f(x,y))$.

(63) Plot the function $f(x) = \frac{3x^2-2x}{x^3+1}$ near $x = 1$.

(64) Plot the following function for various values of a, b.

$$f(x) = \begin{cases} \frac{x^2-4}{x-2} & \text{if } x < 2, \\ ax^2 - bx + 3 & \text{if } 2 \le x < 3, \\ 2x - a + b & \text{if } x \ge 3 \end{cases}$$

if continuous on $\mathbb{R}$. Is $f(x)$ differentiable on $\mathbb{R}$? Explain.

(65) Plot the function

$$g(x,y) = \frac{x^3}{x^2 - y^2}.$$

(66) Let $f(x) = \sinh(x-3) + x^3 - x^4 + 15$. Implement the bisection method on the interval $[-1,3]$ as a sequence of intervals a , b using "If",

(67) Newton's method gives a sequence $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ of hopefully better approximations of α. Write a simple program to implement Newton's method with the function $f(x) = \sinh(x-3) + x^3 - x^4 + 15$.

(68) Let $f(x)$ be the function given in (4). Define a function $g(x) = f(x) - 18$.
(a) If possible, use Newton's method to find some complex solutions of $g(x) = 0$.

(69) Let $g(x)$ be the function given in (3). Plot the surface $z = g(x)$ and the plane $y = 1$.

(70) Plot the functions

$$f(x) = \begin{cases} x^4 - x + 18 & \text{if } x > -1, \\ -x^3 + 2x + 21 & \text{if } x \leq -1. \end{cases}$$

and $f'(x)$.

(71) Try out the following and then modify it to compose your own song. Export as an mp3.

```
song = Play[(2 + Cos[20*t]) * Sin[3100*t + 2 Sin[50*t] ], {t, 0, 2}]
Export["mysong.mp3", song]
```

(72) Try out the following and then explain what the functions do.

```
S = Table[Sin[2*Pi*500*t] + (2 + Cos[20*t]) , {t, 0, 1, 1./2000}];
T = SampledSoundList[S, 2000];
Sound[T]
```

(73) From Blackboard, copy the files entitled 'It's your birthday' by Monk Turner and 'Happy birthday to you' by Sorin Urzica and place these into the Documents folder. Try the following and listen patiently:

```
A = Import["mt.mp3"];
U = AudioData[A];
m = 12000;
V = SampledSoundList[U, m];
Y = Sound[V];
Export["haapppyy.mp3", Y]
```

(74) You can also perform other mathematical transformations on the data such as reversing the data and playing the result. Try the following.

```
A = Import["mt.mp3"];
U = AudioData[A];
X = U[[1]];
Y = Reverse[X];
V = SampledSoundList[Y, 25000];
Z = Sound[V]
```

(75) You can add two sound tracks together. Try the following.

```
Q = Import["su.mp3"];
w = AudioData[A];
P = w[[1]];
mc = Length[P];
uu = Max[mz, mc];
RF = PadRight[X, uu];
SF = PadRight[P, uu];
WK = RF + SF;
V = SampledSoundList[WK, 25000];
Z = Sound[V]
Export["happysum.mp3", Z]
```

(76) Try:

```
Speak["i am so hungry, i could eat a whole cow and not spoil my dinner"]
```

(77) Use the functions "Inverse" and "Determinant" to find the inverses and determinants of the matrices

$$A = \begin{pmatrix} 8 & 1 \\ 2 & 3 \end{pmatrix}, \qquad B = \begin{pmatrix} 3 & 2 \\ 6 & 4 \end{pmatrix},$$

if they exist.

(78) Construct and display, via "MatrixForm[ ]", the block matrix $Q = \begin{pmatrix} A & B \\ B & A \end{pmatrix}$, transpose Q, and then use "ArrayFlatten[ ]" to form a 4×4 non-block matrix from Q.

(79) Let L be the 1×6 array consisting of the first 6 prime numbers. Form a 2×3 matrix M by partitioning the array into two disjoint rows. Now calculate AM or MA, which ever is possible, where A is given in (1).

(80) Find the image of the banana on blackboard, place in the documents folder and

```
A = Import["banana.jpg"]
```

(81) We are able to extract the pixel information from the image and display the first two rows as follows:

```
B = ImageData[A];
Take[B, 2]
```

(82) Use the "Length[ ]" function to determine the dimensions of the image setting the number of rows as m and the number of columns as n.

(83) Use the indexing "B[[i, j, 1]]" to produce an $m \times n$ matrix $M = [b_{ij}]$, where b_{ij} consists of the second entry of the RGB triple at the i, j pixel.

```
M = Table[B[[i, j, 2]], {i, 1, m}, {j, 1, n}];
```

(84) Transpose the matrix M and display the result as an image using "Image".

(85) Construct a list L of points consisting of the $(x, y) = (i, j)$ such that the entry b_{ij} of M satisfies $0.8 < b_{ij} < 0.95$. Use the function "ListPlot" to plot these points in the plane.

```
L = DeleteCases[
   Partition[
    Flatten[Table[
      If[0.8 < M[[i, j]] < 0.95, {i, j}, {0, 0}],
      {i, 1, m}, {j, 1, n}]], 2], {0, 0}];
ListPlot[L, PlotStyle -> PointSize[0.02]]
```

(86) Compute the means $\overline{x}$ and $\overline{y}$ of the x and y coordinates of the data in L, then calculate $T = \{\{x_j, y_j\} - \{\overline{x}, \overline{y}\}\}$.

(87) Construct a matrix

$$X = \begin{pmatrix} x_1 & x_2 & 0 & x_q \\ y_1 & y_2 & 0 & y_q \end{pmatrix}$$

whose columns are the points of T.

(88) Calculate $Y = XX^T$.

(89) Use the function "Eigenvectors[ ]" to calculate the eigenvectors of Y.

(90) Use "ListPlot[ ]" and "Arrow[ ]" to plot the eigenvectors and the points of T in the same image. What do you notice?

(91) Download a different image and repeat the above 11 steps.

(92) Differentiate the following functions using *Mathematica*. For example, to differentiate $f(x) = x^2$ with respect to x, put

```
D[x^2, x]
```

(a) $f(x) = 3x^4 - 16x^3 + 18x^2$,
(b) $f(x) = x^3 - 9x^2 + 30x$,
(c) $f(x) = x^4 - 2x^3 - 2x^2$,
(d) $f(x) = x^5$,
(e) $f(x) = x^4 + x^3$,
(f) $f(x) = \cos(x) + x$,
(g) $f(x) = \sin(2x) - x$,
(h) $f(x) = \log(x) - x^3$,
(i) $f(x) = 2^x - x$,

(93) Find the general solution to the following ODEs using *Mathematica's* DSolve[ ] function, e.g.

```
DSolve[y'[x] + y[x] == 3 Sin[x], y[x], x]
```

would solve the ODE $\frac{dy}{dx} + y = 3\sin(x)$ for y as a function of x.
(a) $\frac{dy}{dx} = 2x^3 - x$,
(b) $\frac{dy}{dx} = \sin(x) - 12x$.

(94) Use *Mathematica's* DSolve[ ] function to solve the following initial value problems, e.g. to solve the IVP $\frac{dy}{dx} + y = 3\sin(x)$, $y(0) = 0$, put

```
DSolve[{y'[x] + y[x] == 3 Sin[x], y[0] == 0}, y[x], x]
```

and plot each of the resulting solutions.
(a) $\frac{dy}{dx} = x^2 + 3$, $y(0) = -2$,
(b) $\frac{dy}{dx} = x + 2\cos(x)$, $y(0) = 3$.

(95) Use the function Integrate[, x] to solve the above (four) ODEs.

(96) Use *Mathematica* to locate and classify the local and global maxima and minima for the following functions $f(x)$ and domains D via taking first and second derivatives. For example, to do this for the function $f(x) = x^3 - 5x + 6$ of $D = [-3, 3]$, put

```
f = x^3 - 5 x + 6;
d = {-3, 3};
Plot[f, {x, d[[1]], d[[2]]}]
g = D[f, x]
S = NSolve[g == 0, x]
n = Length[S];
(* The second derivative *)
h = D[D[f, x], x]
(* The critical points of f *)
Table[ReplaceAll[{S[[j, 1, 2]], f}, S[[j, 1]]], {j, 1, n}]
(* The second derivative test *)
Table[If[ReplaceAll[h, S[[j, 1]]] > 0, min,
  If[ReplaceAll[h, S[[j, 1]]] < 0, max, inconcl]], {j, 1, n}]
```

(a) $f(x) = 3x^4 - 16x^3 + 18x^2$, $D = [-1, 4]$,
(b) $f(x) = x^3 - 9x^2 + 30x$, $D = [-4, 6]$,
(c) $f(x) = x^4 - 2x^3 - 2x^2$, $D = [-1, 3]$,
(d) $f(x) = x^5$, $D = [-2, 2]$,
(e) $f(x) = x^4 + x^3$, $D = [-3, 3]$,
(f) $f(x) = \cos(x) + x$, $D = [-2\pi, 2\pi]$,
(g) $f(x) = \sin(2x) - x$, $D = [-2\pi, 2\pi]$,
(h) $f(x) = \log(x) - x^3$, $D = [0, 3\pi]$,
(i) $f(x) = 2^x - x$, $D = [-5, 5\pi]$.

(97) Use *Mathematica* to calculate the derivative $f'(x) = \frac{dy}{dx}$ for each of the following functions $f(x)$ using the appropriate rule. However in addition to calculating with D[f, x], set out the details of each part of the calculation. For example, if we are to compute $f'(x)$, where $f(x) = \left(x^3 + \cos(x)\right)^5$, we would use the chain rule. You will verify the result in two ways:

```
u = x^3 + Cos[x];
f = u^5 ;
a = D[u, x]
b = D[f, u]
(* Chain rule: *)
a*b
(* f'(x) : *)
D[f, x]
```

For example, if we are to compute $f'(x)$, where $f(x) = \left(x^3 + \cos(x)\right)\sin(x)$, we would use the product rule. You will verify the result in two ways:

```
u = x^3 + Cos[x];
v = Sin[x];
f = u*v ;
a = D[u, x]
b = D[v, x]
(* Product rule: *)
v*a+u*b
```

```
(* f'(x) : *)
D[f, x]
```

(a) $f(x) = x^3 + 3x$,
(b) $f(x) = x^3 \cos\left(x^4 + 2\right)$,
(c) $f(x) = \frac{1}{(5x-3)^2}$,
(d) $f(x) = \frac{1}{\sqrt{9-x^2}}$,
(e) $f(x) = \cos(x)\sin(x)$,
(f) $f(x) = (3x+1)\left(3x^2 + 2x\right)^3$,
(g) $f(x) = x^3 \log(x)$,
(h) $f(x) = xe^x$,
(i) $f(x) = \frac{x+2}{x^2+x}$,
(j) $f(x) = \frac{1}{x^2-16}$.

(98) Use *Mathematica's* built in functions Plot[], Integrate[ ], and NIntegrate[ ] to plot the functions and evaluate the following definite and indefinite integrals:
(a) $\int x^3 + 3x dx$,
(b) $\int x^3 \cos\left(x^4 + 2\right) dx$,
(c) $\int_1^2 \frac{dx}{(5x-3)^2}$,
(d) $\int \frac{dx}{\sqrt{9-x^2}}$,
(e) $\int \cos(x)\sin(x)dx$,
(f) $\int (3x+1)\left(3x^2 + 2x\right)^3 dx$,
(g) $\int x^3 \log(x) dx$,
(h) $\int xe^x dx$,
(i) $\int \frac{x+2}{x^2+x} dx$,
(j) $\int \frac{dx}{x^2-16} dx$.

(99) Evaluate each of the following double integrals using *Mathematica's* Integrate[ ] function. For example, to compute $\int\int x^2 \cos(y) + y dx dy$,

```
f = x^2 Cos[y] + y;
Integrate[Integrate[f, x], y]
```

Use ReplaceAll[ ] to make the conversion to polar coordinates where appropriate.
(a) $\int\int x^2y^2 + 3xy - 7x dx dy$,
(b) $\int_0^3 \int_{-2}^{-1} x^2y^2 + 3xy - 7x dx dy$,
(c) $\int_{-2}^{-1} \int_0^3 x^2y^2 + 3xy - 7x dy dx$,
(d) $\int\int \left(x^2 + y^2\right)^4 dx dy$,
(e) $\int\int \left(x^2 + y^2\right)^4 dx dy$ in polar coordinates $(x, y) = (r\cos(\theta), r\sin(\theta))$,
(f) $\int_0^1 \int_0^1 \left(x^2 + y^2\right)^4 dx dy$ in polar coordinates $(x, y) = (r\cos(\theta), r\sin(\theta))$.

(100) Calculate the improper integral $\int_0^\infty e^{-3x} dx$ by first calculating $\int e^{-3x} dx$ in *Mathematica* and then by using the function Limit[ ]. For example,

```
Limit[(1 + x/n)^n, n -> Infinity]
```

(101) Calculate the Maclaurin series for $\cos(x)$ and determine the radius of convergence using the function Series[ ].

(102) Let $z = f(x) = x^2y^2 + 3xy - 7x$. Calculate $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ using D[f, x] and D[f, y].

(103) Let $z = \sin(x)\cos\left(x^2\right) + ye^x$. Find the tangent plane approximation at $(x, y) = (\pi/2, 1)$. Use

```
Plot3D[{f, g} , {x, -Pi , Pi}, {y, 0, 2}]
```

to show the tangent plane $Z = g(x, y)$ with the function $z = f(x, y)$.

(104) Calculate the following using *Mathematica*. If you do not know the name of the appropriate function,look in the help file.

(a) Let $u = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$ and $v = \begin{pmatrix} 4 \\ 7 \end{pmatrix}$. Calculate the norms of u and v and the angle between u and v.

(b) Let $f(x, y) = 4x^2\cos(y) + 3xy^2$. Calculate the gradient vector ∇f at $(x, y) = (7, 12)$.

(c) Let $f(x, y) = 4x^2\cos(y) + 3xy^2$. Calculate the slope of $f(x, y)$ at $(-4, 5)$ in the direction of the vector $u = (2, -3)$. Plot the function $f(x, y)$ in 3D with Arrow[ ] to illustrate the vector u.

(d) Let $f(x, y) = 4x^2\cos(y) + 3xy^2$. Compute $\frac{\partial^2 f}{\partial x \partial y}$, $\frac{\partial^2 f}{\partial y \partial x}$, $\frac{\partial^2 f}{\partial x^2}$, and $\frac{\partial^2 f}{\partial y^2}$. Use this to calculate $f_{xy}(1, 2)$.

(e) Let $f(x, y, z) = x^3\sin(xz) + z\cos(y) - 15x^2(y+1)z^3$. In which 3 component direction (a vector) is the function $f(x, y, z)$ increasing most rapidly from the point $(2, -1, 5)$? Plot the cross section $z = 5$ in 3D together with a vector pointing the direction in which the function is increasing most rapidly from the point $(2, -1, 5)$.