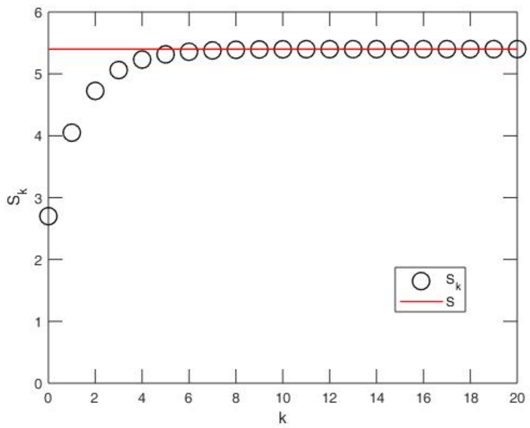
1. Consider a sequence of numbers of the form $x_n = r^n$, where $n = 0, 1, 2, ...$ and $r \in \mathbb{R}$.

   (i) Set $S = \sum_{n=0}^{\infty} \alpha x_n$. For what values of r does the series S converges? Find S and prove your answer.
   S is a Geometric series. Thus, S is convergent for $|r| < 1$.
   Proof:
   First we need to find general formula for partial sum S_n. We can write S_n as:
   $S_n = \alpha + \alpha r + \alpha r^2 + \alpha r^3 + ... + \alpha r^n$, OR
   $rS_n = r\alpha + \alpha r^2 + \alpha r^3 + \alpha r^4 + ... + \alpha r^{n+1}$
   If we subtract these two series:
   $rS_n - S_n = \alpha r^{n+1} - \alpha$
   $S_n(r-1) = \alpha(r^{n+1} - 1)$
   $S_n = \frac{\alpha(r^{n+1}-1)}{(r-1)}$ for $r \neq 1$.
   Then, we take $\lim_{n\to\infty} S_n = ?$
   1) If $r = 1$: $\lim_{n\to\infty} S_n$: does not exist (DNE). $\to S$ diverges.
   2) If $r = -1$: $S_n = \alpha \sum_{k=0}^{n} (-1)^k$. $\lim_{n\to\infty} S_n$: DNE. $\to S$ diverges.
   3) If $|r| < 1$: $\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{\alpha(r^{n+1}-1)}{(r-1)} = \frac{\alpha}{1-r}$. Series S converges to $\frac{\alpha}{1-r}$. 4) IF $|r| > 1$: $\lim_{n\to\infty} S_n$: DNE $\to S$ is divergent.

   (ii) For what values of r does the series S diverge?
   From (i), S is divergent for $|r| \geq 1$.

   (iii) Assume that $r = \frac{1}{2}$ and $\alpha = e$, where e is the Euler's number. Generate a plot that has two curves. Your first curve will depict partial sums S_k versus k where $k = 0, ..., 20$. Show S_k using black circle symbols. Your second curve will depict the sum S for which the series converges when $k \to \infty$. Show this S value as a red solid curve. Label x-axis of your plot as k. Label y-axis of your plot as S_k. Put inset in your plot showing that the circles corresponds to S_k and the red curve corresponds to S. In up to two sentences, explain the convergence of the curve.
   The following curve shows that $S = \sum_{n=0}^{\infty} e(\frac{1}{2})^n$ converges to $2e$.



2. Consider series

$$S = \sum_{n=2}^{\infty} \frac{e^n}{3^{n+1}}$$

   Is the series S convergent or divergent? If you think the series diverges, explain your reason. If you think the series is convergent, then find the sum S.

This is a Geometric series. Note that the lower bound of sum starts from 2. Recalling Theorem 21 in Unit 5, we first need to find the of constant a (first term) and the common ratio of the series. To find a:

$$S = \sum_{n=2}^{\infty} \frac{e^n}{3^{n+1}} = \sum_{n=2}^{\infty} \frac{e^n}{3 \times 3^n} = \sum_{n=2}^{\infty} \frac{1}{3} (\frac{e}{3})^n$$

Now we have the series in the form of Theorem 21. So, the first term of series is $a = e^2/27$ for $n = 2$. And the common ratio is $e/3$ which is in the convergence regime. Thus S converges to:

$$S = \frac{e^2/27}{1 - e/3}$$

3. Use the methods for determining multivariate limits and prove that the following limits do not exist:

(i)

$$\lim_{(x,y)\to(0,0)} \frac{xy^3 \text{cos}x}{2x^2 + y^6}$$

There are two independent variables x and y in this limit. So, we will check the limit in a 2D region:
1) Approach (0, 0) along x-axis ($y = 0$):

$$\lim_{x\to 0} \frac{0}{2x^2} = 0$$

2) Approach (0, 0) along y-axis ($x = 0$):

$$\lim_{y\to 0} \frac{0}{y^6} = 0$$

3) Now lets take a path in 2D such that the degrees of variables in nominator and denominator match up. One way is to eliminate the variable with smaller power. So, let's take path $x = y^3$. After substituting, the only independent variable is y:

$$\lim_{y\to 0} \frac{y^3 y^3 \text{cos} y^3}{2y^6 + y^6} = \lim_{y\to 0} \frac{y^6 \text{cos} y^3}{3y^6} = \lim_{y\to 0} \frac{\text{cos} y^3}{3} = \frac{1}{3}$$

So, travelling along x-axis, y-axis, and $x = y^3$, the value of function is different (i.e., $0 \neq \frac{1}{3}$). Thus, the limit DNE at (0, 0).

(ii)

$$\lim_{(x,y,z)\to(0,0,0)} \frac{xy + yz + xz}{x^2 + y^2 + z^2}$$

There are three independent variables in this limit. So, we will check the limit in a 3D region:
1) Approach (0, 0, 0) along x-axis ($y = 0, z = 0$):

$$\lim_{x\to 0} \frac{0}{x^2} = 0$$

We can repeat this along y-axis and z-axis, and the limit will be zero. So, we can test travelling along a 3D path such that is parametric and the degrees of nominator and denominator match up. For example, let $x = t$, $y = t$, $z = t$ which is line in 3D region. Now approaching (0, 0, 0) along this line is equivalent to approach 0 with the new independent variable t. After substituting:

$$\lim_{t\to 0} \frac{t^2+t^2+t^2}{t^2+t^2+t^2} = \lim_{t\to 0} \frac{3t^2}{3t^2} = 1$$

We see that as we travel along x-axis and the line $x = y = z$, the value of function is different (i.e., $0 \neq 1$). Thus, the limit DNE at (0, 0, 0).

4. Consider the following function

$$f(x) = \frac{1}{x^2 - x}$$

on interval (0, 1).

(i) Find the critical points of $f(x)$.

$$f'(x) = \frac{1-2x}{(x^2-x)^2}$$

To check the critical points:

$$f'(x) = 0 \to 1 - 2x = 0 \to x = 1/2$$

$x = 1/2$ is within the interval. Thus, at critical point $x = 1/2$, $f(1/2) = -4$.

(ii) What are the global maximum and minimum of $f(x)$, if exist?
We now need to evaluate the function at the boundaries of the interval. Since the interval is open, we find one-sided limits:

$$\lim_{x\to 0^+} \frac{1}{x^2 - x} = -\infty$$

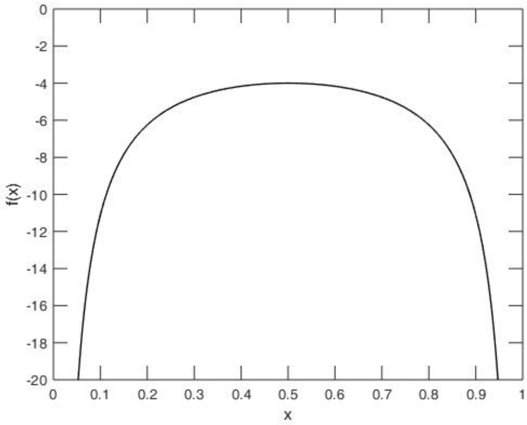
and

$$\lim_{x\to 1^-} \frac{1}{x^2 - x} = -\infty$$

These two limits show that there is no global minimum. From (i), the global maximum is $f(1/2) = -4$.

(iii) Where do the global maximum and minimum of $f(x)$ occur, if exist?
From (i), the global maximum -4 occurs at $x = 1/2$.

(iv) Plot $f(x)$ versus x. Show the curve as a black solid curve. Label x-axis of your plot as x. Label y-axis of your plot as $f(x)$. Do your findings of global maximum, global minimum, and critical points agree with your plot? Explain it in up to three sentences.


0
-2
-4
-6
-8
-10
-12
-14
-16
-18
-20
f(x)
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
x