1. Consider the $m \times n$ matrix A and the $n \times p$ matrix B, with respective entries for each matrix represented via $a_{ij} = i - j$ and $b_{ij} = i + j$.
   (a) Assume $m = 2$, $n = 2$, and $p = 4$. Evaluate the matrix $C = AB$.
   (b) Derive (prove) the formula,
   $$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}.$$
   (c) Use Mathematica to obtain a formula for $\sum_{k=1}^{n} k^2$.
   (d) Assume now arbitrary m, n, and p values. Use the above formulas to obtain a general formula for the entries of C, c_{ij}. Check that your results agree with (a). You may use Mathematica (or Wolfram alpha) to simplify expressions if needed.

2. Find two $n \times n$ matrices A and B such that $AB \neq BA$.

3. Find two 2×2 matrices A and B, with all entries nonzero, such that $AB = BA$.

4. For any angle $\theta \in [0, 2\pi]$, consider the rotation matrix,
   $$A_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$
   (a) Compute the determinant $|A_\theta|$.
   (b) Prove that for any two angles θ_1 and θ_2, $A_{\theta_1} A_{\theta_2} = A_{\theta_2} A_{\theta_1}$.
   (c) Given some A_θ consider the inverse matrix and represent it via A_η where $\eta \in [0, 2\pi]$.

5. Prove that $(AB)^T = B^T A^T$.

6. Consider 10^6 random 2×2 matrices with entries that are uniformly and identically drawn from the range $[0, 1]$. For each such matrix denote the determinant via D_i with $i = 1, \ldots, 10^6$.
   (a) What is the minimal possible value that D_i may take? How about the maximal value?
   (b) Reason about the expected (average) value. What is it?
   (c) Use Mathematica (or other software) to generate $D_1, \ldots, D_{10^6}$ and plot a histogram of the distribution.

7. Consider the numbers 1,2,3,. . . ,9 organized in a 3×3 matrix, each number appearing exactly once. Find a configuration of these numbers in the matrix that has the maximal possible determinant and prove this is the case using a Mathematica (or other language) computer program, or using some other means.

8. Consider the vector $[1 \;\; 2 \;\; 3 \;\; \ldots \;\; n]^T$ and the vector $[-1 \;\; -2 \;\; -3 \;\; \ldots \;\; -n]^T$. Find an expression for the inner product of these vectors. You may use results from previous exercises for this. Also, determine the norm of each of these vectors and explain why it is the same.

9. Consider the matrix A and the vector y given by,
   $$A = \begin{bmatrix} 1 & 2.9 \\ 1 & 4.3 \\ 1 & 5.2 \\ 1 & 6.9 \\ 1 & 8.3 \end{bmatrix} \quad \text{and} \quad y = \begin{bmatrix} 3 \\ 2.9 \\ 5.3 \\ 7.8 \\ 5.5 \end{bmatrix}.$$
   Generate a million random entries in $[0, 5] \times [0, 5]$ of the vector $\beta = [\beta_0 \;\; \beta_1]^T$ searching for a vector that approximates the minimum of $||A\beta - y||$. Then plot the points $(A_{i,2}, y_i)$ for $i = 1, \ldots, 5$ against a line $y = \beta_0 + \beta_1 x$ where β_0 and β_1 are the (approximate) minimum.