Math7501 Quiz 2, Semester 1, 2020 - Sample Answers

1. Consider the universal set, $U = \{x \in \mathbb{Z}^+ : x \leq 10\}$ and the sets $A = \{x \in U : x > 7\}$ and $B = \{1, 2, 3\}$

   (a) Write out all the elements of U explicitly

   $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

   (b) Write out all of the elements of A explicitly

   $A = \{8, 9, 10\}$

   (c) What is $A \cap B$?

   $\emptyset$

   (d) What is $A \cup B$?

   $\{1, 2, 3, 8, 9, 10\}$

   (e) What is $A^c \cap B$?

   $A^c = \{1, 2, 3, 4, 5, 6, 7\}$
   $\therefore A^c \cap B = \{1, 2, 3\} = B$

   (f) What is $A \times B$?

   $\{(8, 1), (8, 2), (8, 3), (9, 1), (9, 2), (9, 3), (10, 1), (10, 2), (10, 3)\}$

   (g) Write the elements of $\mathcal{P}(\mathcal{B})$?

   $\{\emptyset,\{1\},\{2\},\{3\},\{1,2\},\{1,3\},\{2,3\},\{1,2,3\}\}$

   (h) What is $|\mathcal{P}(\mathcal{B})|$?

   $8 = 2^{|B|}$

2. Consider the following logical expression $(A \lor B) \land \neg (A \land B)$

   (a) Write the truth table for the above expression

A	B	$A \lor B$	$A \land B$	$\neg (A \land B)$	$(A \lor B) \land \neg (A \land B)$
T	T	T	T	F	F
T	F	T	F	T	T
F	T	T	F	T	T
F	F	F	F	T	F

   (b) Write an expression using only ANDs, ORs and NOTs that is logically equivalent to the above expression

   Lots of possible answers! It's equivalent to an XOR operation. Say $\neg\neg ((A \lor B) \land \neg (A \land B))$, or $(A \land \neg B) \lor (\neg A \land B)$. Verify with a truth table to show equivalence, e.g.

A	B	$A \land \neg B$	$\neg A \land B$	$(A \land \neg B) \lor (\neg A \land B)$	
T	T	F	F	F	
T	F	T	F	T	
F	T	F	T	T	
F	F	F	F	F	

3. Prove $\sum_{i=1}^{n} (2i - 1) = n^2$

   Show true for $n = 1$

   $\sum_{i=1}^{1} (2i - 1) = 1 = (1)^2 \therefore true$

   Assume true for k, i.e.

   $\sum_{i=1}^{k} (2i - 1) = k^2$

   Show true for $k + 1$

   $\sum_{i=1}^{k+1} (2i - 1) = \sum_{i=1}^{k} (2i - 1) + (2(k + 1) - 1)$
   $= k^2 + (2(k + 1) - 1)$
   $= k^2 + 2k + 2 - 1$
   $= k^2 + 2k + 1$
   $= (k + 1)^2$

   Therefore by the principle of mathematical induction the result is true $(n \geq 1)$

4. Consider the exponential function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = e^x$

(a) what is the domain of f?

$\mathbb{R}$

(b) what is the codomain of f?

$\mathbb{R}$

(c) what is the range of f?

$\mathbb{R}^+ = \{y \in \mathbb{R} : y > 0\}$

(d) is f one to one (injective)?

Yes, since for the exponential function is continuous and monotonic, i.e. if $x < y$ then $f(x) < f(y)$, so the only way $f(x) = f(y)$ is if $x = y$.

(e) is f onto (surjective)?

No. For example for there exsits no x such that $f(x) = -1$

(f) is f invertible? If it is not invertible what is a simple redefinition of f that will make it invertible?

Needs to be injective and surjective (bijection) to be invertible, so the answer is no. However if change the codomain to equal the range it becomes invertible, i.e.$f : \mathbb{R} \to \mathbb{R}^+$ defined by $f(x) = e^x$