This is the cover sheet. Write your name and good luck.

Instructions:

The number of points per question is marked next to the question.

The total number of points is 100.

Answer all questions on this exam paper.

(1) Assume you have a grayscale image represented in a 50×200 matrix A where A_{ij} represents the pixel intensity of the matrix as a number in the range $[0,1]$, with $A_{ij} = 0$ implying dark and $A_{ij} = 1$ bright.

Let $\mathbf{1}_n$ denote a column vector of n 1's and set,

$$m = \mathbf{1}_{50}^T \, A \, \mathbf{1}_{200}.$$

(1a) Represent the minimizer of,

$$L(x) = \sum_{i=1}^{50} \sum_{j=1}^{200} (x - A_{ij})^2,$$

in terms of m.
Comment: The minimizer is the value of x that minimizes $L(x)$.

(15 points)

Solution:

Seek $\frac{dL(x)}{dx} = 0$:

$$\frac{dL(x)}{dx} = \sum_{i=1}^{50} \sum_{j=1}^{200} 2(x - A_{ij}) = 0.$$

Hence

$$\sum_{i=1}^{50} \sum_{j=1}^{200} (x - A_{ij}) = 0,$$

or,

$$10^4 x = \sum_{i=1}^{50} \sum_{j=1}^{200} A_{ij} = m.$$

Hence,

$$x^* = 10^{-4} m.$$

Can also check it is a minimizer via $\frac{dL(x)^2}{dx^2} > 0$:

$$\frac{dL(x)^2}{dx^2} = 2 \times 10^4 > 0.$$

Note that a purley algebraic solution is possible because $L(x)$ is just a parabola in x of the form,

$$L(x) = ax^2 + bx + c.$$

□

(1b) Say that for demonstration purposes, you wish to find the minimizer of $L(x)$ as defined above via a one dimensional gradient descent algorithm. For this you set $x_0 = 0.5$, and then for $n \geq 1$, you proceed via the recursion:

$$x_{n+1} = x_n - \eta L'(x_n),$$

where L' is derivative of L. Here, η, the learning rate, is a positive value. Establish a range for η over which the sequence $\{x_n\}_{n=1}^{\infty}$ converges to your solution of (1a).

(20 points)

Solution:

Using the derivative from the previous solution,

$$\begin{aligned} x_{n+1} &= x_n - \eta \sum_{i=1}^{50} \sum_{j=1}^{200} 2(x_n - A_{ij}) \\ &= x_n - 2 \times 10^4 \, \eta \, x_n + 2 \, \eta \, m \\ &= (1 - 2 \times 10^4 \, \eta) x_n + 2 \, \eta \, m \end{aligned}$$

This is of the form,

$$x_{n+1} = a x_n + b,$$

and hence,

$$\begin{aligned} x_1 &= a x_0 + b \\ x_2 &= a(a x_0 + b) + b = a^2 x_0 + ab + b \\ x_3 &= a(a^2 x_0 + ab + b) + b = a^3 x_0 + a^2 b + ab + b = a^3 x_0 + b \sum_{i=0}^{2} a^i \\ &\ldots \\ x_n &= a^n x_0 + b \sum_{i=0}^{n-1} a^i = a^n x_0 + b \frac{a^n - 1}{a - 1}. \end{aligned}$$

Hence if $|a| < 1$,

$$\lim_{n \to \infty} x_n = \frac{b}{1 - a},$$

and otherwise $\{x_n\}$ diverges. In our case $a = 1 - 2 \times 10^4 \eta$ and if $|a| < 1$ the limit is the answer from the previous question:

$$\frac{2 \eta m}{1 - (1 - 2 \times 10^4 \eta)} = 10^{-4} m.$$

Now $a < 1$ so $|a| < 1$ means that $-1 < a$ or $-2 < -2 \times 10^4 \eta$ or,

$$\eta < 10^{-4}.$$

□

(2) Let γ be a positive quantity. Consider the probability density function of a distribution defined over $[0, \infty)$:

$$f(x) = K\frac{x}{\gamma}e^{-\frac{x^2}{\gamma}},$$

where K is some positive constant (depending on γ).

(2a) Determine K such that the integral,

$$\int_0^\infty f(x)\,dx = 1.$$

(15 points)

Solution:

This is the Rayleigh distribution. Set $u = x^2$ and hence $du/2 = x\,dx$ and hence,

$$\int K\frac{x}{\gamma}e^{-x^2/\gamma}dx = \int \frac{K}{2\gamma}e^{-u/\gamma}du = -\frac{K}{2}e^{-u/\gamma} + C = -\frac{K}{2}e^{-x^2/\gamma} + C.$$

Hence for the improper integral,

$$\int_0^\infty f(x)\,dx = \lim_{x\to\infty} -\frac{K}{2}e^{-x^2/\gamma} + \frac{K}{2} = \frac{K}{2}.$$

Hence $K = 2$.
Note: compare to the density appearing in Wikepdia with $\gamma = 2\sigma^2$.
□

(2b) The median is the value $\mu > 0$ such that,

$$\int_0^\mu f(x)\,dx = \frac{1}{2}.$$

Find the median in terms of γ.

(15 points)

Solution:

Following the indefinite integral and the value of K from the previous question,

$$\int_0^u f(x) = -e^{-x^2/\gamma}\Big|_{x=\mu} + e^{-x^2/\gamma}\Big|_{x=0} = 1 - e^{-\mu^2/\gamma} = 1/2.$$

Or,

$$\log 1/2 = -\mu^2/\gamma.$$

Or

$$\mu = \sqrt{\gamma \log 2}.$$

□

(3) Let θ be a real value parameter and consider the matrix $B_\theta(x)$ represented as a function of x and parameterized by θ:

$$B_\theta(x) = \begin{bmatrix} x^2 & 0 \\ e^{\cos(x)} & e^{\theta x} \end{bmatrix}.$$

You now wish to calculate the definite improper integral

$$M_\theta = \int_{-10}^{\infty} |B_\theta(u)|\, du.$$

where $|\cdot|$ represents the determinant. Determine the set of values of θ for which the integral converges.

(15 points)

Solution:

First observe that $|B_\theta(x)| = x^2 e^{\theta x}$. We can then compute the integral using integration by parts twice to get and explicit expression for

$$\int u^2 e^{\theta u}\, du,$$

but this is not necessarily needed. Instead we can reason that only if $\theta < 0$ the integral converges and otherwise not. Hence the solution is $\theta \in (-\infty, 0)$.
□

(4) Assume $\lambda > 0$ and let $f(x) = \lambda e^{-\lambda x}$. Now for $k = 0, 1, 2, \ldots$, define,

$$M_k = \int_0^\infty x^k f(x)\, dx.$$

Prove that,

$$M_k = \frac{k!}{\lambda^k}.$$

(20 points)

Solution:

(Note: This is the k'th moment of an exponential distribution).
Look at, $J_n \equiv \int x^n \lambda e^{-\lambda x}\, dx$ for $n \geq 1$. and carry out integration by parts:

$$u = x^n, \qquad u' = nx^{n-1}, \qquad v' = \lambda e^{-\lambda x}, \qquad v = -e^{-\lambda x}.$$

Hece,

$$J_n = \int uv'\, dx = uv - \int u'v\, dx = -x^n e^{-\lambda x} + \frac{n}{\lambda} \int x^{n-1} \lambda e^{-\lambda x} dx = -x^n e^{-\lambda x} + \frac{n}{\lambda} J_{n-1}.$$

Writing this recursion for the definite improper integral works just the same:

$$M_n = -x^n e^{-\lambda x} \Big|_0^\infty + \frac{n}{\lambda} M_{n-1} = \frac{n}{\lambda} M_{n-1}.$$

Now $M_0 = 1$ by basic integration of $\lambda e^{-\lambda x}$ and hence the result follows by induction.
□

END OF EXAMINATION