You can use Mathematica as an aid for many of the computations, however make sure to do hand calculations where suitable as well.

1. Consider the 2×4 matrix,
$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix}.$$
Consider also the 4×2 matrices
$$L = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad \text{and} \qquad R = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$
   (a) Denote $B = AL$ and $C = AR$. Describe in words the matrices B and C in terms of the matrix A.
   (b) Determine the matrix $G = AA^T$.
   (c) Determine the matrix $H = A^T A$.
   (d) Is the matrix G symmetric? How about the matrix H? Would your answer always be true if G and H were made up of other values of A or is it specific to the values of A presented in this question?
   (e) Compute the determinant of G and the determinant of H.
   (f) One of the two matrices G and H is invertible. Which one? What is the inverse?

2. Consider the $m \times n$ matrix A and the $n \times p$ matrix B, with respective entries for each matrix represented via $a_{ij} = i - 2j$ and $b_{ij} = 2i + j$.
   (a) Assume $m = 3$, $n = 2$, and $p = 4$. Evaluate the matrix $C = AB$.
   (b) Derive (prove) the formula,
$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}.$$
   (c) Use Mathematica to obtain a formula for $\sum_{k=1}^{n} k^2$.
   (d) Assume now arbitrary m, n, and p values. Use the above formulas to obtain a general formula for the entries of C, c_{ij}. Check that your results agree with (a). You may use Mathematica to simplify expressions if needed.

3. In general matrix multiplication is not commutative, that is $AB \neq BA$.
   (a) Find two $n \times n$ matrices A and B such that $AB \neq BA$.
   (b) Find two 2×2 matrices A and B, with all entries nonzero, such that $AB = BA$.
   (c) Consider now random 3×3 matrices where the entries are drawn uniformly from the set $\{1, 2, \ldots, r\}$. Write a Mathematica program that estimates the probability of $AB = BA$ for two random matrices for values of $r = 2, 3, 4, 5$. Do this for each r by simulating (drawing random) $100,000$ pairs of matrices A and B and checking if they commute each time. Then report the proportion.

4. For any angle $\theta \in [0, 2\pi]$, consider the rotation matrix,
$$A_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$
   (a) Compute the determinant $|A_\theta|$.

(b) Prove that for any two angles θ_1 and θ_2, $A_{\theta_1} A_{\theta_2} = A_{\theta_2} A_{\theta_1}$.

(c) Given some A_θ consider the inverse matrix and represent it via A_η where $\eta \in [0, 2\pi]$.

5. Consider the matrix A and the vector y given by,

$$A = \begin{bmatrix} 1 & 2.4 \\ 1 & 4.7 \\ 1 & 4.9 \\ 1 & 2.9 \\ 1 & 8.1 \end{bmatrix} \qquad \text{and} \qquad y = \begin{bmatrix} 3.1 \\ 2.7 \\ 4.8 \\ 7.6 \\ 5.4 \end{bmatrix}.$$

(a) Generate a million random entries in $[0, 5] \times [0, 5]$ of the vector $\beta = [\beta_0 \;\; \beta_1]^T$ searching for a vector that approximates the minimum of $||A\beta - y||$. Then plot the points $(A_{i,2}, y_i)$ for $i = 1, \ldots, 5$ against a line $y = \beta_0 + \beta_1 x$ where β_0 and β_1 are the (approximate) minimum.

(b) The theory of least squares (studied in future courses e.g. MATH7502) indicates that the β that minimizes $||A\beta - y||$ (as well as $||A\beta - y||^2$) is given by,

$$\beta = (A^T A)^{-1} A^T y,$$

as long as the matrix $A^T A$ is non-singular. Use this formula to find the best β and compare it to the β you found in question 1.

6. Consider the sets $A = \{k \in \mathbb{N} \mid k \text{ is divisible by } 3\}$ and $B = \{k \in \mathbb{N} \mid k{+}1 \text{ is divisible by } 3\}$.

(a) What is the set $A \cap B$?

(b) What is the set $\mathbb{N} \setminus (A \cup B)$?

(c) Consider the additional set,

$$C_r = \{k \in \mathbb{N} \mid k \le r\},$$

where r is a number. Write Mathematica code to determine $|A \cap C_r|$ and try it for $r = 10, 20, 100$.

7. Consider the set $B = \{a, b, c, \{c\}, \emptyset\}$. Determine if each statement is true or false and explain why:

(a) $|2^B| = 2^{|B|}$.

(b) $\{b, c\} \subset B$.

(c) $\{b, c\} \in 2^B$

(d) $\{a\} \in B$.

(e) $\emptyset \subset B$.

(f) $\emptyset \in B$.

8. Consider the equation,

$$x_1 + x_2 + \ldots + x_n = k,$$

where n and k are integers and the solution to the equation, denoted $(x_1, \ldots, x_n)$, is restricted for the set of non-negative integers.

(a) Show that the number of solutions is,

$$\binom{n + k - 1}{k}.$$

(b) Write a Mathematica program that presents all solutions and demonstrate it working for $k = 2$ and $n = 6$.

(c) Consider playing monopoly and rolling two (indistinguishable) dice. How many possible outcomes are there? What is the relationship to the above items?