(1) Consider the 2×2 matrix A with entries $A_{ij} = \binom{i+j}{i}$.

(i) Determine the numerical values for the entries of A.

(ii) Determine $\det(A)$.

(iii) If it exists, determine A^{-1} otherwise indicate that the inverse does not exist.

(iv) Consider the (column) vector $\mathbf{1}$ which has entries 1 and 1. Denote $u = A\mathbf{1}$. Determine $||u||$.

(v) Take now an $n \times n$ matrix A with entries $A_{ij} = \binom{n}{i}$ and let e_i be the n-unit vector ($[e_i]_j = 0$ for $j \neq i$ and $[e_i]_j = 1$ for $j = i$). Find a simple expression for,

$$\sum_{i=1}^{n} e_i^T A \, e_i.$$

(2) Let the set $\mathcal{M}$ be the set of all 2×2 matrices with real elements in the range $[-1, 1]$. Let the set $\mathcal{N}$ be the set of all 2×2 matrices with elements that are in the set $\{-1, +1\}$. Let I be the 2×2 identity matrix.

(i) Is $\Big(I \in \mathcal{M}\Big) \wedge \Big(\mathcal{N} \subset \mathcal{M}\Big)$ true or false?

(ii) Determine the value of $\Big|\mathcal{N}\Big|$.

(iii) Determine the value of $\Big|2^{\mathcal{N}}\Big|$.

(iv) Find an element $X \in \mathcal{N}$ such that $\det(X) = 2$.

(v) Consider the (column) vector $\mathbf{1}$ which has entries 1 and 1. Now for any 2×2 matrix X denote $\overline{X}$ as $\mathbf{1}^T X \mathbf{1}$. What is the maximal value of $\overline{X}$ that can be obtained if considering all $X \in \mathcal{N}$?

(vi) Does the same answer hold if considering all $X \in \mathcal{M}$? Briefly explain.

(vii) Is $\Big(\mathcal{M} \setminus \mathcal{N} = \emptyset\Big) \Rightarrow \Big(I \notin \mathcal{M}\Big)$ true or false?

(viii) Determine the value of $\Big|\mathcal{N} \times \mathcal{N}\Big|$.

(3) Prove by induction or any other means:

$$\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}.$$