SOLUTION

(1) Consider the 2×2 matrix A with entries $A_{ij} = \binom{i+j}{i}$.

(i) Determine the numerical values for the entries of A.

$$A = \begin{bmatrix} \binom{2}{1} & \binom{3}{1} \\ \binom{3}{2} & \binom{4}{2} \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 3 & 6 \end{bmatrix}.$$

(ii) Determine $\det(A)$.

$$\det(A) = 2 \times 6 - 3 \times 3 = 3.$$

(iii) If it exists, determine A^{-1} otherwise indicate that the inverse does not exist.

$$A^{-1} = \frac{1}{3} \begin{bmatrix} 6 & -3 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 2/3 \end{bmatrix}.$$

(iv) Consider the (column) vector $\mathbf{1}$ which has entries 1 and 1. Denote $u = A\mathbf{1}$. Determine $||u||$.

$u = [5 \;\; 9]^T$. Hence, $||u|| = \sqrt{5^2 + 9^2} = \sqrt{106} \approx 10.3$.

(v) Take now an $n \times n$ matrix A with entries $A_{ij} = \binom{n}{i}$ and let e_i be the n-unit vector ($[e_i]_j = 0$ for $j \neq i$ and $[e_i]_j = 1$ for $j = i$). Find a simple expression for,

$$\sum_{i=1}^{n} e_i^T A \, e_i.$$

We have that, $e_i^T A \, e_i. = A_{ii} = \binom{n}{i}$. Hence,

$$\sum_{i=1}^{n} e_i^T A \, e_i = \sum_{i=1}^{n} \binom{n}{i} = .\Big(\sum_{i=0}^{n} \binom{n}{i} \Big) - \binom{n}{0} = 2^n - 1.$$

(2) Let the set $\mathcal{M}$ be the set of all 2×2 matrices with real elements in the range $[-1, 1]$. Let the set $\mathcal{N}$ be the set of all 2×2 matrices with elements that are in the set $\{-1, +1\}$. Let I be the 2×2 identity matrix.

(i) Is $\Big(I \in \mathcal{M}\Big) \wedge \Big(\mathcal{N} \subset \mathcal{M}\Big)$ true or false?

It is true that $I \in \mathcal{M}$. It is also true that any element (matrix) of $\mathcal{N}$ is an element of $\mathcal{M}$ and hence $\mathcal{N} \subset \mathcal{M}$. Hence the conjunction ($\wedge$) is also true.

(ii) Determine the value of $\Big|\mathcal{N}\Big|$.

Each matrix has 4 entries. Each entry has 2 options. Hence there are $2^4 = 16$ matrices.

(iii) Determine the value of $\Big|2^{\mathcal{N}}\Big|$.

$|2^{\mathcal{N}}| = 2^{|\mathcal{N}|} = 2^{16} = 65,536$.

(iv) Find an element $X \in \mathcal{N}$ such that $\det(X) = 2$.

Take for example,

$$X = \begin{bmatrix} +1 & -1 \\ 1 & +1 \end{bmatrix}.$$

(v) Consider the (column) vector $\mathbf{1}$ which has entries 1 and 1. Now for any 2×2 matrix X denote $\overline{X}$ as $\mathbf{1}^T X \mathbf{1}$. What is the maximal value of $\overline{X}$ that can be obtained if considering all $X \in \mathcal{N}$?

We have that $\overline{X}$ is a scalar with,

$$\overline{X} = \mathbf{1}^T X \mathbf{1} = \sum_{i=1}^{2} \sum_{j=1}^{n} A_{ij}.$$

The maximal value is when $A_{ij} = +1$ for all i and j and thus the maximal value of $\overline{X}$ is 4.

(vi) Does the same answer hold if considering all $X \in \mathcal{M}$? Briefly explain.

Yes. Since the maximal A_{ij} for elements of $\mathcal{N}$ is the same as the maximal A_{ij} for elements of $\mathcal{M}$, even when considering the (infinite) set $\mathcal{M}$, the maximal $\overline{X}$ is 4.

(vii) Is $\Big(\mathcal{M} \setminus \mathcal{N} = \emptyset\Big) \Rightarrow \Big(I \notin \mathcal{M}\Big)$ true or false?

Observe that $\mathcal{M} \setminus \mathcal{N} \neq \emptyset$ because there are many elements of $\mathcal{M}$ that are not in $\mathcal{N}$. Hence as a logical statement we have,

$$\text{false} \quad \Rightarrow \quad \text{something.}$$

In this case, "something" is false because $I \in \mathcal{M}$, but that doesn't matter. This statement is true because $F \Rightarrow$ something is true.

(viii) Determine the value of $\Big|\mathcal{N} \times \mathcal{N}\Big|$.

We have that if sets $\mathcal{A}$ and $\mathcal{B}$ are finite then $|\mathcal{A} \times \mathcal{B}| = |\mathcal{A}|\,|\mathcal{B}|$. Hence the answer is $16 \times 16 = 256$.

(3) Prove by induction or any other means:

$$\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}.$$

Proof by induction:
Let us show the statement is true for $n = 1$:

$$1^3 = \frac{1^2(1+1)^2}{4}.$$

Now assume the statement is true for n and we shall show it is true for $n + 1$:

$$\begin{aligned}
\sum_{k=1}^{n+1} k^3 &= \sum_{k=1}^{n} k^3 + (n+1)^3 \\
&= \frac{n^2(n+1)^2}{4} + (n+1)^3 \\
&= \frac{(n+1)^2(n^2+4(n+1))}{4} \\
&= \frac{(n+1)^2((n+1)+1)^2}{4}.
\end{aligned}$$

This is exactly the statement for $n + 1$ and this completes the induction proof.