(1) Evaluate each of the following expressions. Fully justify your use of any theorems from class.

(i) $\lim_{x\to\infty} e^{-\frac{1}{x^2}}$.

The exponential function is continuous, and $e(0) = 1$. Since $\lim_{x\to\infty} -\frac{1}{x^2} = 0$, we conclude that $\lim_{x\to\infty} e^{-\frac{1}{x^2}} = 1$.

(ii) $\lim_{x\to 2} \frac{x-2}{\cos(\pi x)+1-x}$.

As x approaches 2, both the numerator and the denominator approach 0. The derivative of the numerator is 1, and the derivative of the denominator is $-\pi\sin(\pi x) - 1$, which is non-zero close to $x = 2$. Now $\lim_{x\to 2} \frac{1}{-\pi\sin(\pi x)-1} = -1$, so by l'Hôpital's rule, we have $\lim_{x\to 2} \frac{x-2}{\cos(\pi x)+1-x} = -1$.

(iii) $\lim_{x\to 0} f(x)$, where $f(x)$ is any real function such that $x < f(x) < -x$ for all $x < 0$, and $-x^{10} < f(x) < x^5$ for all $x > 0$.

Define $h(x) = \begin{cases} x^5 \text{ if } x > 0 \\ -x \text{ if } x < 0, \end{cases}$ and also define $g(x) = \begin{cases} -x^{10} \text{ if } x > 0 \\ x \text{ if } x < 0. \end{cases}$ Then $g(x) < f(x) < h(x)$ for all $x \neq 0$. Since $\lim_{x\to 0} h(x) = \lim_{x\to 0} g(x) = 0$, the squeeze theorem implies $\lim_{x\to 0} f(x) = 0$.

(iv) $\sum_{n=1}^{\infty} \frac{e}{\pi^n}$.

The formula for geometric series implies that the answer is $\frac{e}{\pi-1}$. It is important to note that $\pi > 1$ for this formula to apply.

(v) $\lim_{x\to\infty} \frac{x-2}{\cos(\pi x)+1-x}$.
Hint: It is useful to note that for $x > 10$, $-x \leq \cos(\pi x) + 1 - x \leq 2 - x$.

Therefore, $\frac{x-2}{2-x} \leq \frac{x-2}{\cos(\pi x)+1-x} \leq \frac{x-2}{-x}$ (paying close attention to rules for inequalties when fractions are involved.) The squeeze Theorem then gives that the answer is -1.

(2) Consider the function

$$f(x) = \begin{cases} 0 & \text{if } x = 0, \\ e^{-\frac{1}{x^2}} & \text{if } x \neq 0. \end{cases}$$

(i) Show that f is continuous at any non-zero value of x.

The exponential function is continuous anywhere, and $-\frac{1}{x^2}$ is continuous for all non-zero values of x. Therefore, $e^{-\frac{1}{x^2}}$ is continuous for all non-zero values of x.

(ii) Calculate $f'(x)$ for each $x \neq 0$.

The derivative of the exponential function is itself, and the derivative of $-\frac{1}{x^2}$ is $\frac{2}{x^3}$. Therefore, by the chain rule, $f'(x) = \frac{2}{x^3}e^{-\frac{1}{x^2}}$

(iii) Show that f is continuous at the point $x = 0$.

We have $f(0) = 0$ by definition. Therefore, it suffices to show that $\lim_{x\to 0} f(x) = 0$. This follows because $\lim_{x\to 0} -\frac{1}{x^2} = -\infty$, and $\lim_{y\to -\infty} e^{-y} = 0$.

(3) Consider the real-valued function

$$f(x) = x^3 + 4x^2 - 20x + 1.$$

(i) Calculate $f'(x), f''(x)$

A straightforward application of the differentiation rules for polynomials gives $f'(x) = 3x^2 + 8x - 20$ and $f''(x) = 6x + 8$.

(ii) Find all of the critical points of f; determine if they are local maxima/minima.

Using the quadratic formula to find the roots of $f'(x)$, we find that the critical points are $x = -\frac{4}{3} - \frac{2\sqrt{19}}{3}$.

Since the leading term in $f'(x)$ is $3x^2$, and $3 > 0$, the function $f'(x)$ must be negative between these two critical points, and positive everywhere else. Therefore, the first derivative test implies that the first of these points $(-\frac{4}{3} + \frac{2\sqrt{19}}{3})$ is a local maximum, and the second is a local minimum.

(iii) How many distinct real valued roots does f have? Prove your answer by making reference to the IVT and the MVT.

Note that f, f' are continuous everywhere. Therefore, when using the IVT and MVT, we will no longer reference this fact.

Let $x_1 = -\frac{4}{3} - \frac{2\sqrt{19}}{3}$ and $x_2 = -\frac{4}{3} + \frac{2\sqrt{19}}{3}$ be the two critical points. Using a calculator, we find that $f(x_1) > 0$ and $f(x_2) < 0$.

The IVT then implies that there is at least one root on (x_1, x_2). The MVT implies that there is no more than one root because the derivative is negative here.

Now pick two other values $x_0 = -10$ and $x_3 = 10$ (there are plenty of other choices that can work). We calculate $f(x_0) < 0$ and $f(x_3) > 0$. The IVT then implies the existence of two more roots. Since $f' > 0$ outside $[x_1, x_2]$, the MVT implies that there are no more roots.