You can use Mathematica as an aid for many of the computations, however make sure to do hand calculations where suitable as well.
Note: The Mathematica based solution are in `https://github.com/yoninazarathy/MATH7501-2021/blob/master/Assignment1Sol/Sol1Mathematica.pdf`

1. Consider the 2×4 matrix,
$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix}.$$
Consider also the 4×2 matrices
$$L = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad \text{and} \qquad R = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

(a) Denote $B = AL$ and $C = AR$. Describe in words the matrices B and C in terms of the matrix A.
Solution: The matrix B is made up of the first two columns of the matrix A and the matrix C is made up of the last two columns of the matrix A.

(b) Determine the matrix $G = AA^T$.
Solution:
$$G = \begin{bmatrix} 30 & 70 \\ 70 & 174 \end{bmatrix}.$$

(c) Determine the matrix $H = A^T A$.
Solution:
$$H = \begin{bmatrix} 26 & 32 & 38 & 44 \\ 32 & 40 & 48 & 56 \\ 38 & 48 & 58 & 68 \\ 44 & 56 & 68 & 80 \end{bmatrix}.$$

(d) Is the matrix G symmetric? How about the matrix H? Would your answer always be true if G and H were made up of other values of A or is it specific to the values of A presented in this question?
Solution: Yes both are symmetric ($G = G^T$ and $H = H^T$). Let's prove this in general (for any matrix A):
$$G^T = \big(AA^T\big)^T = (A^T)^T A^T = AA^T = G.$$
Similarly for H.

(e) Compute the determinant of G and the determinant of H.
Solution: $\det(G) = 30 \times 174 - 70 \times 70 = 320$.
For $\det(H)$ more work is needed in a recursive manual calculation. However, using further results of linear algebra (see for example the next course MATH7502), it is clear that the matrix $A^T A$ is singular (non-invertible) because the columns of A are not linearly independent (they cannot be linearly independent as there are four columns, each of length < 4). Hence $\det(H) = 0$.

(f) One of the two matrices G and H is invertible. Which one? What is the inverse?
Solution: The determinant of G is not 0 and hence it is invertible (the inverse exists). As opposed to that H is singular (not invertible). This is the inverse of G.
$$G^{-1} = \frac{1}{320} \begin{bmatrix} 174 & -70 \\ -70 & 30 \end{bmatrix}.$$

2. Consider the $m \times n$ matrix A and the $n \times p$ matrix B, with respective entries for each matrix represented via $a_{ij} = i - 2j$ and $b_{ij} = 2i + j$.

   (a) Assume $m = 3$, $n = 2$, and $p = 4$. Evaluate the matrix $C = AB$.
   Solution:

$$A = \begin{bmatrix} -1 & -3 \\ 0 & -2 \\ 1 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 3 & 4 & 5 & 6 \\ 5 & 6 & 7 & 8 \end{bmatrix}.$$

   Hence just multiplying we have,

$$C = AB = \begin{bmatrix} -18 & -22 & -26 & -30 \\ -10 & -12 & -14 & -16 \\ -2 & -2 & -2 & -2 \end{bmatrix}.$$

   (b) Derive (prove) the formula,

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}.$$

   Solution: Here is one proof (not the most popular) (there are many more proofs): Sketch an $n \times n$ upper-diagonal matrix. The sum $\sum_{k=1}^{n} k$ is the number of (potentially) non-zero entries (these are the entries on the diagonal and above the diagonal). This is because the first column will have 1 entry, the second 2 entries, the third 3 entries, up to the n'th column which has n entries.
   Now compute how many entries are there in general. Take an $n \times n$ matrix excluding the diagonal so there are $n^2 - n$ entries. Divide this by 2 to get the number of entries above the diagonal, namely,

$$\frac{n^2 - n}{2}.$$

   Now add to this the n entries in the diagonal to get,

$$\frac{n^2 - n}{2} + n = \frac{n(n+1)}{2}.$$

   This proves the result.

   (c) Use Mathematica to obtain a formula for $\sum_{k=1}^{n} k^2$.
   Solution: See Mathematica file. We get:

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}.$$

   If you wanted to, one way to prove it is via induction. Alternatively (if you wanted to - although not needed to get points for this question), to derive it directly one way is to consider,

$$(k-1)^3 = k^3 - 3k^2 + 3k - 1,$$

   or

$$k^3 - (k-1)^3 = 3k^2 - 3k + 1.$$

   Now the sum on the left is a telescopic sum so when summing the equation on $k = 1, \dots, n$ we remain on the left hand side with n^3. Hence the summation of the equation (also using the summation of of the arithmetic sum from above):

$$n^3 = 3\sum_{k=1}^{n} k^2 - 3\frac{n(n+1)}{2} + n.$$

   This allows to isolate $\sum_{k=1}^{n} k^2$ to get the desired result.

(d) Assume now arbitrary m, n, and p values. Use the above formulas to obtain a general formula for the entries of C, c_{ij}. Check that your results agree with (a). You may use Mathematica to simplify expressions if needed.

Solution: We have that c_{ij} is the inner product of the i'th row of A and the j'th column of B. That is,

$$\begin{aligned} c_{ij} &= \sum_{k=1}^{n} A_{ik}B_{kj} \\ &= \sum_{k=1}^{n}(i-2k)(2k+j) \\ &= \sum_{k=1}^{n}\big(2ik + ij - 4k^2 - 2jk\big) \\ &= 2(i-j)\big(\sum_{k=1}^{n} k\big) - 4\big(\sum_{k=1}^{n} k^2\big) + nij \\ &= 2(i-j)\frac{n(n+1)}{2} - 4\frac{n(n+1)(2n+1)}{6} + nij. \end{aligned}$$

This expression doesn't simplify beyond this, but there are different forms which can be used. See the Mathematica file for a check that the formula agrees with the result of (a).

3. In general matrix multiplication is not commutative, that is $AB \neq BA$.

(a) Find two $n \times n$ matrices A and B such that $AB \neq BA$.

Solution: There are infinitely many options one example is A being the matrix with entries 1,2,3, and 4. And B the matrix with all 1 entries.

(b) Find two 2×2 matrices A and B, with all entries nonzero, such that $AB = BA$.

Solution: The way the question is worded, there wasn't a requirement that $A \neq B$. Hence you could have any A (with nonzero entries) and then set B to be A and that would work. But say we requested different matrices. In this case you can have $B = \alpha A$ where α is some non-zero constant. You can also find A and B that are "completely different" and commute. In a more advanced (Linear Algebra) course you may study when two matrices commute. Another example to mention is the rotation matrices of the next question. These always commute.

(c) Consider now random 3×3 matrices where the entries are drawn uniformly from the set $\{1, 2, \ldots, r\}$. Write a Mathematica program that estimates the probability of $AB = BA$ for two random matrices for values of $r = 2, 3, 4, 5$. Do this for each r by simulating (drawing random) $100,000$ pairs of matrices A and B and checking if they commute each time. Then report the proportion.

Solution: See Mathematica file. For $r = 2$ there is a chance of about 0.0033 of getting a commuting pair, but for higher r the chance quickly goes to 0.

4. For any angle $\theta \in [0, 2\pi]$, consider the rotation matrix,

$$A_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

(a) Compute the determinant $|A_\theta|$.

Solution:

$$\det(A_\theta) = \cos^2\theta + \sin^2\theta = 1.$$

(b) Prove that for any two angles θ_1 and θ_2, $A_{\theta_1}A_{\theta_2} = A_{\theta_2}A_{\theta_1}$.
Solution:

$$
\begin{aligned}
A_{\theta_1}A_{\theta_2} &= \begin{bmatrix}\cos\theta_1 & -\sin\theta_1\\ \sin\theta_1 & \cos\theta_1\end{bmatrix}\begin{bmatrix}\cos\theta_2 & -\sin\theta_2\\ \sin\theta_2 & \cos\theta_2\end{bmatrix}\\
&= \begin{bmatrix}\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 & -(\cos\theta_1\sin\theta_2 + \cos\theta_2\sin\theta_1)\\ \cos\theta_1\sin\theta_2 + \cos\theta_2\sin\theta_1 & \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2\end{bmatrix}\\
&= \begin{bmatrix}\cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2)\\ \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2)\end{bmatrix}\\
&= A_{\theta_1+\theta_2}.
\end{aligned}
$$

The simplification above used trigonometric identities for the cosine and sum of angles.

(c) Given some A_θ consider the inverse matrix and represent it via A_η where $\eta \in [0, 2\pi]$.
Solution: Observe that since this is a rotation matrix, with have that A_0 (no rotation) is the identity matrix. Using the above we get that,

$$A_\theta^{-1} = A_{-\theta}.$$

5. Consider the matrix A and the vector y given by,

$$A = \begin{bmatrix}1 & 2.4\\ 1 & 4.7\\ 1 & 4.9\\ 1 & 2.9\\ 1 & 8.1\end{bmatrix} \qquad \text{and} \qquad y = \begin{bmatrix}3.1\\ 2.7\\ 4.8\\ 7.6\\ 5.4\end{bmatrix}.$$

(a) Generate a million random entries in $[0,5]\times[0,5]$ of the vector $\beta = [\beta_0 \;\; \beta_1]^T$ searching for a vector that approximates the minimum of $||A\beta - y||$. Then plot the points $(A_{i,2}, y_i)$ for $i = 1,\ldots,5$ against a line $y = \beta_0 + \beta_1 x$ where β_0 and β_1 are the (approximate) minimum.
Solution: See Mathematica file.

(b) The theory of least squares (studied in future courses e.g. MATH7502) indicates that the β that minimizes $||A\beta - y||$ (as well as $||A\beta - y||^2$) is given by,

$$\beta = (A^T A)^{-1} A^T y,$$

as long as the matrix $A^T A$ is non-singular. Use this formula to find the best β and compare it to the β you found in question 1.
Solution: See Mathematica file.

6. Consider the sets $A = \{k \in \mathbb{N} \mid k \text{ is divisible by } 3\}$ and $B = \{k \in \mathbb{N} \mid k+1 \text{ is divisible by } 3\}$.

(a) What is the set $A \cap B$?
Solution: $A \cap B = \emptyset$ because you can't have to consecutive numbers k and $k+1$ both be divisible by 3.

(b) What is the set $\mathbb{N} \setminus (A \cup B)$?
Solution: $A \cup B = \{3, 6, 9, \ldots\} \cup \{2, 5, 8, \ldots\}$. Hence,

$$\mathbb{N} \setminus (A \cup B) = \{1, 4, 7, 10, \ldots\}.$$

(c) Consider the additional set,

$$C_r = \{k \in \mathbb{N} \mid k \le r\},$$

where r is a number. Write Mathematica code to determine $|A \cap C_r|$ and try it for $r = 10, 20, 100$.
Solution: See Mathematica file.

7. Consider the set $B = \{a, b, c, \{c\}, \emptyset\}$. Determine if each statement is true or false and explain why:

   (a) $|2^B| = 2^{|B|}$.
   Solution: True. The number of elements in the power set (number of subsets) is two to the size of the set. It is 32.

   (b) $\{b, c\} \subset B$.
   Solution: True. Every element of $\{b, c\}$ is an element of B.

   (c) $\{b, c\} \in 2^B$
   Solution: True. This statement is equivalent to $\{b, c\} \subset B$.

   (d) $\{a\} \in B$.
   Solution: False. There is no element in B that is $\{a\}$.

   (e) $\emptyset \subset B$.
   Solution: True. The empty set is a subset of any set.

   (f) $\emptyset \in B$.
   Solution: True. The specific set B has the empty set as an element.

8. Consider the equation,

$$x_1 + x_2 + \ldots + x_n = k,$$

where n and k are integers and the solution to the equation, denoted $(x_1, \ldots, x_n)$, is restricted for the set of non-negative integers.

   (a) Show that the number of solutions is,

$$\binom{n+k-1}{k}.$$

   Solution: This problem can be recast as the problem of having k balls (right hand side of the equation) and placing them in the n bins, $x_1,\ldots,x_n$. The balls are indistinguishable and there is not a limit of the number of balls per bin. In this case, each representation of a balls into bins arrangement can be cast as a sequence of balls (of which there are k) and bin separators of which there are $n-1$. Hence we need to choose out of $n-1+k$ characters (balls or separators) which are the balls (k).

   (b) Write a Mathematica program that presents all solutions and demonstrate it working for $k = 2$ and $n = 6$.
   Solution: See Mathematica file.

   (c) Consider playing monopoly and rolling two (indistinguishable) dice. How many possible outcomes are there? What is the relationship to the above items?
   Solution: In this case $k = 2$ and $n = 6$ The number of outcomes is then,

$$\binom{6+2-1}{2} = \binom{7}{2} = \frac{7!}{5!2!} = 21.$$