MATH7501 Assignment Solutions

These solutions produced by Mitchell Griggs.
(a) 
\[
\lim_{x \to 0^+} \left( \frac{1}{x} - \frac{1}{\sin x} \right) = \lim_{x \to 0^+} \left( \frac{\sin(x) - x}{x \sin x} \right) = \lim_{x \to 0^+} \frac{-x^3}{3!} + \frac{x^5}{5!} - \ldots = 0.
\]

(b) 
\[
\lim_{x \to \pi/4} (1 - \tan(x))(\sec(2x)) = \lim_{x \to \pi/4} \frac{\cos(x) - \sin(x)}{\cos(x) \cos(2x)} = \lim_{x \to \pi/4} \left( \frac{1}{\cos(x)} \right) \lim_{x \to \pi/4} \frac{\cos(x) - \sin(x)}{\cos(2x)} = \sqrt{2} \lim_{x \to \pi/4} \frac{-\sin(x) - \cos(x)}{-2 \sin(2x)} = \sqrt{2} \cdot \frac{1}{\sqrt{2}} = 1.
\]

(c) 
\[
\lim_{x \to \infty} x \left( 2^{1/x} - 1 \right) = \lim_{t \to 0^+} \frac{1}{t} \left( 2^t - 1 \right) = \lim_{t \to 0^+} \frac{2^t \log(2)}{1} = \log(2).
\]

(d) 
\[
\lim_{x \to 1^{1/x}} \exp \left( \frac{1}{1 - x} \log(x) \right) = \exp \left( \lim_{x \to 1} \frac{\log(x)}{1 - x} \right) = \exp \lim_{x \to 1} \left( \frac{1}{x \cdot (-1)} \right) = e^{-1}.
\]

(e) For continuity, we need 
\[
2(1) - (1)^2 = (1)^2 + k(1) + p,
\]
which rearranges to give
\[
k = -p.
\]
For differentiability, we also need

\[ 2 - 2(1) = 2(1) + k, \]

so \( k = -2 \), and then \( p = -k = 2 \).

(f) We need

\[ (1)^2 + k(1) + p \geq 1, \]

which rearranges to give \( k + p \geq 0 \), and we also need the positive gradient

\[ 2x + k > 0, \text{ for all } x > 1, \]

so \( k > -2 \). Notice that \( f \) is already increasing when \( x < 1 \).

(g) The function \( f \) is the sum of continuous functions \((x \mapsto \sin(x)/x) \) is continuous when \( x \neq 0 \), so is also continuous on \( \mathbb{R} \setminus \{0\} \).

\[
f(1) = 1 + \sin(1) - 3 < 0 \quad \text{and} \quad
f(3) = 27 + \frac{\sin(3)}{3} - 3 > 27 + \frac{-1}{3} - 3 > 17,
\]

so by the MVT (Mean-Value Theorem), there exists \( \alpha \in (1, 3) \subseteq \mathbb{R} \) satisfying \( f(\alpha) = 17 \). To find \( \alpha \) where \( f(\alpha) = 0 \), consider the midpoint of \( (1, 3) \):

\[
f(2) = 8 + \frac{\sin(2)}{2} - 3 = 5 - \frac{\sin(2)}{2} > 0.
\]

Consider the midpoint of \( (1, 2) \):

\[
f(1.5) > 0.
\]

Consider the midpoint of \( (1, 1.5) \):

\[
f(1.25) < 0.
\]

The next midpoint is 1.375. Some students may be content with approximating \( \alpha \approx 1.375 \), but some may continue the method further, increasing the accuracy of this approximation.

Continuing in this manner, we eventually conclude \( \alpha \approx 1.31 \).
Showing that the limit has different values along any two lines is sufficient. We give three examples in this solution.

Approaching along the line $y = 3x^2$ gives

$$f(x, y) = \frac{1 - e^0}{2x^2 + 9x^4} \to 0.$$  

Approaching along $x = 0$ and $y \to 0^+$ gives

$$f(x, y) = \frac{1 - e^{-y}}{y^4} \to \infty.$$  

Approaching along $x = 0$ and $y \to 0^-$ gives

$$f(x, y) \to -\infty.$$  

(i) $M^{-1}$ exists if, and only if, $ad - bc \neq 0$, and is given by

$$M^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$  

$$MM^{-1} = \frac{1}{ad - bc} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I.$$  

(j)

$$A^{-1} = \frac{1}{22} \begin{pmatrix} 3 & -1 \\ -2 & 8 \end{pmatrix}$$

and $B^{-1}$ does not exist since $3 \cdot 4 - 2 \cdot 6 = 12 - 12 = 0$.

(k)

$$(A + B)^T = [a_{ij} + b_{ij}]^T = [a_{ji} + b_{ji}] = [a_{ji}] + [b_{ji}] = [a_{ij}]^T + [b_{ij}]^T$$
\[ (AB)^T = [a_{i1}b_{ij} + a_{i2}b_{2j}]^T = [a_{i1}b_{ij1} + a_{i2}b_{ij2}] = [b_{j1}a_{i1} + b_{j2}a_{2i}] = [b_{ji}][a_{ji}] = B^TA^T. \]

With \( C = B + I \), \((AC)^{-1}\) satisfies

\[ (AC)(AC)^{-1} = I \]
\[ A(B + I)(AC)^{-1} = I \]
\[ \Rightarrow (AC)^{-1} = (B + I)^{-1}A^{-1} \]
\[ = \left( \begin{array}{cc} 4 & 2 \\ 6 & 5 \end{array} \right)^{-1} \left( \begin{array}{cc} 3 & -1 \\ 1 & 1 \end{array} \right) \]
\[ = \frac{1}{176} \left( \begin{array}{cc} 19 & -21 \\ -26 & 38 \end{array} \right). \]

(l) \[ AB = \begin{pmatrix} 30 & 20 \\ 24 & 16 \end{pmatrix} \text{ and } \]
\[ BA = \begin{pmatrix} 28 & 9 \\ 56 & 18 \end{pmatrix}. \]

(m) \[
\det(BA) = \det(B)\det(A) = \frac{77}{3} \cdot (-39) = -\frac{-3003}{3} = -1001.
\]

(n) Writing \( A = [a_{ij}] \) and \( B = [b_{ij}] \) \((i, j \in \{1, \ldots, n\})\), the trace of \( A + B \) is

\[ \text{tr}(A + B) = \sum_{k=1}^{n} (a_{kk} + b_{kk}) \]
\[
\sum_{k=1}^{n} a_{kk} + \sum_{k=1}^{n} b_{kk} = \text{tr}(A) + \text{tr}(B).
\]

\((o)\)

\[
\det \begin{pmatrix} 8 - \lambda & 1 \\ 2 & 3 - \lambda \end{pmatrix} = 0,
\]

so

\[
0 = (8 - \lambda)(3 - \lambda) - 2 = 24 - 11\lambda + \lambda^2 - 2 = \lambda^2 - 11\lambda + 22
\]

\[
\Rightarrow \lambda = \frac{11}{2} \pm \frac{\sqrt{33}}{2}.
\]

\((p)\) \(A^t A\) is an \(n \times n\) matrix where each entry is 1, so \(\text{tr}(A^t A) = n\).

\(A A^t\) is a \(1 \times 1\) matrix; \(A A^t = (1)\), so \(\text{tr}(A A^t) = n\).

\((q)\) We have

\[
ABA' = (1 \, \cdots \, 1) \, B \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix},
\]

so \((1, \ldots, 1)B = [c_{1j}]\) is a \(1 \times n\) matrix, with \(j\)th entry

\[
c_{1j} = \sum_{i=1}^{n} b_{ij} = \sum_{i=1}^{n} b_{ij},
\]

so

\[
AB = \begin{pmatrix} \sum_{i=1}^{n} b_{i1}, \ldots, \sum_{i=1}^{n} b_{in} \end{pmatrix},
\]

and therefore

\[
ABA' = \left(\sum_{i=1}^{n} b_{i1}\right) + \cdots + \left(\sum_{i=1}^{n} b_{in}\right) = \sum_{i=1}^{n} \sum_{j=1}^{n} b_{ij}.
\]
Therefore \( b_{ij} = i + j \) gives

\[
ABA' = \left( \sum_{i=1}^{n} (i + 1) \right) + \cdots + \left( \sum_{i=1}^{n} (i + n) \right)
\]

\[
= n + 2n + \cdots + n \cdot n + n \sum_{i=1}^{n} i
\]

\[
= n \left( \sum_{i=1}^{n} i + \sum_{i=1}^{n} i \right)
\]

\[
= 2n \sum_{i=1}^{n} i
\]

\[
= 2n \frac{n(n + 1)}{2}
\]

\[
= n^2(n + 1).
\]

\((r)\) With \( b_{ij} = i + j^2 \),

\[
ABA' = \sum_{j=1}^{n} \sum_{i=1}^{n} b_{ij}
\]

\[
= \sum_{j=1}^{n} \sum_{i=1}^{n} (i + j^2)
\]

\[
= \sum_{j=1}^{n} \left( nj^2 + \sum_{i=1}^{n} i \right)
\]

\[
= \sum_{j=1}^{n} \left( nj^2 + \frac{n(n + 1)}{2} \right)
\]

\[
= \frac{n^2(n + 1)}{2} + n \sum_{j=1}^{n} j^2
\]

\[
= \frac{n^2(n + 1)}{2} + \frac{n^2(n + 1)(2n + 1)}{6}.
\]

\((s)\) Consider \( A = I \) and \( B = -I \).