Assignment 2

MATH 7502 - Semsester 2, 2018

Mathematics for Data Science 1

Created by Zhihao Qiao, Maria Kleshnina and Yoni Nazarathy

Question 1

(a) Find the general solutions of the system whose augmented matrix is given by

$$\begin{bmatrix} 1 & -7 & 0 & 6 & 5 \\ 0 & 0 & 1 & -2 & -3 \\ -1 & 7 & -4 & 2 & 7 \end{bmatrix}.$$

(b) Under what condition on b_1, b_2, b_3 is this system solvable?

$$x + 2y - 2z = b_1$$
$$2x + 5y - 4z = b_2$$
$$4x + 9y - 8z = b_3.$$

(c) Give an exmaple of an incosistent undetermined system of two equations with three unknowns.

Solution

(a) The RREF of the argumented matrix is

$$\begin{bmatrix} 1 & -7 & 0 & 6 & 5 \\ 0 & 0 & 1 & -2 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The last row is all zero, therefore we have infinite number of solutions.

$$x_3 - x_4 = -3$$

,

$$x_1 - 7x_2 + 6x_4 = 5$$

we have two free varaibles x_4 and x_2, therefore we have the general solutions

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5 + 7x_2 - 6x_4 \\ x_2 \\ -3 + x_4 \\ x_4 \end{bmatrix} = \begin{bmatrix} 7 \\ 1 \\ 0 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} -6 \\ 0 \\ 1 \\ 1 \end{bmatrix} x_4 + \begin{bmatrix} 5 \\ 0 \\ -3 \\ 0 \end{bmatrix}$$

(b) The rref of the row reduced matrix is

$$\begin{bmatrix} 4 & 9 & -8 & b_3 \\ 0 & 0.5 & 0 & \frac{2b_2 - b_3}{2} \\ 0 & 0 & 0 & \frac{2b_1 + b_2 - b_3}{2} \end{bmatrix}$$

Therefore, it has solution only if

$$2b_1 + b_2 - b_3 = 0$$

.

(c)

$$x_1 + x_2 + x_3 = 1$$
$$x_1 + x_2 + x_3 = 0$$

Question 2

(a) Let $v_1 = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$, $v_2 = \begin{bmatrix} -3 \\ 1 \\ 8 \end{bmatrix}$,and $y = \begin{bmatrix} h \\ -5 \\ -3 \end{bmatrix}$. For what value(s) of h is y in the plane spanned by v_1 and v_2.

(b) Explain why $\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}\}$ is a basis for R^2 (Hint: Write e_1 and e_2 as linear combinations of these vectors.)

(c) Find three different bases for the column space of $U = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}$. Then find two different bases for the row space of U.

Solution

(a) we try to write

$$y = a_1 y_1 + a_2 y_2$$

, therefore we have a set of linear equations

$$a_1 - 3a_2 = h$$
$$a_2 = -5$$
$$-2a_1 + 8a_2 = -3$$

The solutions is $a_1 = -\frac{37}{2}$, therefore, $h = -\frac{7}{2}$

(b)since the base of R^2 is e_1 and e_2, therefore, have

$$e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{2}{3}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

and

$$e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = -\frac{1}{3}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

(c) The row bases are the first row and second row.

Since the first and second row contains a pivot, therefore the column bases are the first and second column.

Question 3

Consider the transofrmation $T : R^3 \to R^3$ that projects vectors onto the line $l = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} . t \in R$

(a) Find a formula for $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} \right)$, and prove that T is a linear transformation.

(b) Find matrix A, asssociated with T.

(c) Determine the relationship between the row space of A row(A) and the line l.

(d) Determine the null space, Null(A).

Solution

(a) if we project a vector onto the line, which is

$$proj_v(\begin{bmatrix} x \\ y \\ z \end{bmatrix}) = \frac{\begin{bmatrix} x \\ y \\ z \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} x+z \\ 0 \\ x+z \end{bmatrix}$$

(b) we want to find a matrix that $T(v) = Av$, which is

$$A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{2} \begin{bmatrix} x+z \\ 0 \\ x+z \end{bmatrix}$$

Obviously,

$$A = \begin{bmatrix} 0.5 & 0 & 0.5 \\ 0 & 0 & 0 \\ 0.5 & 0 & 0.5 \end{bmatrix}$$

(c) The row spance of A has basis $\{1, 0, 1\}$, which is the same as the directional vector of the line l.

(d) The null space of A is that

$$x + z = 0$$

, which is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} y$$

Therefore, the it is the vector space spanned by the basis $\{(1, 0, -1), (0, 1, 0)\}$.

Question 4

(a) Define $A = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$, Find the eigenvalues and associated eigenvectors of A.

(b) Can you represent $\begin{bmatrix} 8 \\ 10 \end{bmatrix}$ as a linear combination of the eigenvectors? If so , do so.

(c) Consider the linear transformation $T : R^2 \to R^2$ where $T(v) = Av$. Suppose n is a postive integer and we write $T^n : R^2 \to R^2$ for the composition of T with itself n times. Use your answer in (a) and (b) to calculate

$$T^{47}\left(\begin{bmatrix} 8 \\ 10 \end{bmatrix}\right)$$

.

Solution

(a) The eigenvalue of A is we have

$$\det(A - \lambda A) = -\lambda + \lambda^2 - 2$$

solving that the eigenvalues are $\lambda_1 = 2, \lambda_2 = -1$.

for $\lambda_1 = 2$,, we have

$$\begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 2 \begin{bmatrix} x \\ y \end{bmatrix}$$

we have

$$y = 2x, 2x + y = 2y$$

the egien vectors are $v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

for $\lambda_2 = -1$,, we have

$$\begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = - \begin{bmatrix} x \\ y \end{bmatrix}$$

we have

$$y = -x, 2x + y = -y$$

the egien vectors are $v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

(b) This is $6v_1 - 2v_2$.

(c) We have

$$T^{47}\left(\begin{bmatrix} 8 \\ 10 \end{bmatrix} \right) = T^{47}(6v_1 - 2v_2) = 6T^{47}(v_1) - 2T^{47}(v_2) = (6)(\lambda_1^{47})v_1 - (2)(\lambda_2^{47})v_2$$

.

Question 5

Suppose we have matrices $A, B, X,$ and Y with $AX = BY$.

(a) Give an example showing that $A \neq 0$ is not enough to conclude that $X = Y$.

(b) Show that if A is left-invertible, we can conlcude from $AX = AY$ that $X = Y$.

(c) Show that if A is not left-invertible, there are matrices X and Y with $X \neq Y$, and $AX = AY$.

Solution

(a) for example if we have A and X are identity matrix, and $Y = B^{-1}$ where B is not an identity matrix..

(b)

$$AX = AY$$
$$A^{-1}AX = A^{-1}AY$$
$$X = Y$$

(c)

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$
$$X = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$Y = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$AX = AY = 0$, but $X \neq Y$.

Question 6

Consider the stacked vectors

$$c_1 = \begin{bmatrix} a_1 \\ b_1 \end{bmatrix}, \ldots .. c_k = \begin{bmatrix} a_k \\ b_k \end{bmatrix}.$$

where $a_1, \ldots, a_k$ are n-vectors and $b_1, \ldots, b_k$ are m-vectors. For each case, either prove or provide a counter example.

(a) Suppose $a_1, \ldots, a_k$ are linearly independent (we make no assumptions about $b_1, \ldots, b_k$). Can we conclude that the stacked vectors $c_1, \ldots, c_k$ are linearly indepndent?

(b) Suppose $a_1, \ldots, a_k$ are linearly dependent (we make no assumptions about $b_1, \ldots, b_k$). Can we conclude that the stacked vectors $c_1, \ldots, c_k$ are linearly depndent?

Solution

(a) Suppose $m_1c_1 + \ldots m_kc_k = 0$, and m_i are real coefficients. This is equivalent that

$$\begin{bmatrix} m_1a_1 + \ldots + m_ka_k \\ m_1b_1 + \ldots + m_kb_k \end{bmatrix} = 0$$

Since a_i are linearly independent, therefore, the first row is true only if $m_1 = m_2 = \ldots = m_k = 0$

Hence it is enough to conlcude the stack vectors are also linearly independent.

(b) Similar to above, if $a_1, \ldots, a_k$ are linearly dependent, but if $b_1, \ldots, b_k$ are linearly independent, then stacked vectors are linearly independent, therefore the second statement is wrong.

Question 7

Let $G \in R^{m \times n}$ represent a contignecy matrix of m students who are members of n groups with

$$G_{ij} = \begin{cases} 1 & \text{student i is in group j} \\ 0 & \text{student i is not in group j} \end{cases}$$

(a) What is the meaning of the 3rd column of G?

(b) What is the meaning of the 2nd row of G?

(c) Give a simple formula for the n-vectors M, where M_i is the total membership in the group i.

(d) Interpret $(GG^T)_{ij}$ in simple Enlgish.

(e) Interpret $(G^T G)_{ij}$ in simple English.

Solution

(a) The third column is G_{i3}, which determines the group members in group 3.

(b) The 2nd row is G_{2j}, which determines which groups does student 2 attend to .

(c)

$$M = [1 \quad 1 \quad \dots \quad 1] \times G$$

. The left has n entries of 1.

The product is the sum of each column, which is the total membership in each group.

(d) $(GG^T)_{ij}$ means how many groups in common of student$i$ and student j attend to .

(e) $(G^T G_{ij})$ means how many students in common does group i and group j have.

Question 8

An n-vector x is symmetric if $x_k = x_{n-k+1}$ for $k = 1, \dots, n.$ It is anti-symmetric if $x_k = -x_{n-k+1}$ for $k = 1, \dots n.$

(a) Show that every vector x can be decomposed in a unique way as sum $x = x_s + x_a$ of a symmetric vector x_s and an anti-symmetric vector x_a.

(b) Find matrices A_s and A_a such that $x_s = A_s x$, and $x_a = A_a x$ for all x.

Solution

(a) To prove this, it is obvious that the sum of symmetric vectors are symmetric, and the sum of anti-symmetric vectors are anti-symmetric.$.

and each vector can be write as an unique decomposition of the elementary bases $e_1, \ldots, e_n$.

Suppose

$$x = a_1e_1 + \ldots a_ne_n$$

if n is even, (WLOG we give a simple example of $n = 4$

$$x = a_1e_1 + a_2e_2 + a_3e_3 + a_4e_4$$

$$= \frac{1}{2}\Bigg((a_1 + a_4)e_1 + (a_1 - a_4)e_1 + (a_2 + a_3)e_2 + (a_2 - a_3)e_2 + (a_3 + a_2)e_3 + (a_3 - a_2)e_3 + ($$

$$= \frac{1}{2}\Bigg((a_1 + a_4)(e_1 + e_4) + (a_2 + a_3)(e_2 + e_3)\Bigg) + \frac{1}{2}\Bigg((a_1 - a_4)e_1 + (a_4 - a_1)e_4 + (a_2 - a_3)e$$

This decomposites a vector into a symmetric and anti-symmetric part and it is unique.

if n is odd, (WLOG we give a simple example of $n = 5$),

$$x = a_1e_1 + a_2e_2 + a_3e_3 + a_4e_4 + 5e_5$$

$$= \frac{1}{2}\Bigg((a_1 + a_5)e_1 + (a_1 - a_5)e_1 + (a_2 + a_4)e_2 + (a_2 - a_4)e_2 + 2e_3 + (a_4 + a_2)e_4 + (a_4 - a_2)$$

$$= \frac{1}{2}\Bigg((a_1 + a_5)(e_1 + e_5) + (a_2 + a_4)(e_2 + e_4) + e_3\Bigg) + \frac{1}{2}\Bigg((a_1 - a_5)e_1 + (a_5 - a_1)e_5 + (a_2 -$$

This decomposite a vector into a symmetric and anti-symmetric part and it is also unique..

(b)