Assignment 2

MATH 7502 - Semsester 2, 2018

Mathematics for Data Science 1

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Question 1

(a) Find the general solutions of the system whose augmented matrix is given by

$$\begin{bmatrix} 1 & -7 & 0 & 6 & 5 \\ 0 & 0 & 1 & -2 & -3 \\ -1 & 7 & -4 & 2 & 7 \end{bmatrix}.$$

(b) Under what condition on b_1, b_2, b_3 is this system solvable?

$$x + 2y - 2z = b_1$$
$$2x + 5y - 4z = b_2$$
$$4x + 9y - 8z = b_3.$$

(c) Give an exmaple of an incosistent undetermined system of two equations with three unknowns.

Question 2

(a) Let $v_1 = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, v_2 = \begin{bmatrix} -3 \\ 1 \\ 8 \end{bmatrix}$,and $y = \begin{bmatrix} h \\ -5 \\ -3 \end{bmatrix}$. For what value(s) of h is y in the plane spanned by v_1 and v_2.

(b) Explain why $\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \}$ is a basis for R^2 (Hint: Write e_1 and e_2 as linear combinations of these vectors.)

(c) Find three different bases for the column space of $U = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}$. Then find two different bases for the row space of U.

Question 3

Consider the transofrmation $T : R^3 \rightarrow R^3$ that projects vectors onto the line $l = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} . t \in R$

(a) Find a formula for $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right)$, and prove that T is a linear transformation.

(b) Find matrix A, asssociated with T.

(c) Determine the relationship between the row space of A row(A) and the line l.

(d) Determine the null space, Null(A).

Question 4

(a) Define $A = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$, Find the eigenvalues and associated eigenvectors of A.

(b) Can you represent $\begin{bmatrix} 8 \\ 10 \end{bmatrix}$ as a linear combination of the eigenvectors? If so , do so.

(c) Consider the linear transformation $T : R^2 \rightarrow R^2$ where $T(v) = Av$. Suppose n is a postive integer and we write $T^n : R^2 \rightarrow R^2$ for the composition of T with itself n times. Use your answer in (a) and (b) to calculate

$$T^{47}\left(\begin{bmatrix} 8 \\ 10 \end{bmatrix}\right)$$

.

Question 5

Suppose we have matrices $A, B, X,$ and Y with $AX = BY$.

(a) Give an example showing that $A \neq 0$ is not enough to conclude that $X = Y$.

(b) Show that if A is left-invertible, we can conlcude from $AX = AY$ that $X = Y$.

(c) Show that if A is not left-invertible, there are matrices X and Y with $X \neq Y$, and $AX = AY$.

Question 6

Consider the stacked vectors

$$c_1 = \begin{bmatrix} a_1 \\ b_1 \end{bmatrix}, \ldots . c_k = \begin{bmatrix} a_k \\ b_k \end{bmatrix}.$$

where $a_1, \ldots, a_k$ are n-vectors and $b_1, \ldots, b_k$ are m-vectors. For each case, either prove or provide a counter example.

(a) Suppose $a_1, \ldots, a_k$ are linearly independent (we make no assumptions about $b_1, \ldots, b_k$). Can we conclude that the stacked vectors $c_1, \ldots, c_k$ are linearly indepndent?

(b) Suppose $a_1, \ldots, a_k$ are linearly dependent (we make no assumptions about $b_1, \ldots, b_k$). Can we conclude that the stacked vectors $c_1, \ldots, c_k$ are linearly depndent?

Question 7

Let $G \in R^{m \times n}$ represent a contignecy matrix of m students who are members of n groups with

$$G_{ij} = \begin{cases} 1 & \text{student i is in group j} \\ 0 & \text{student i is not in group j} \end{cases}$$

(a) What is the meaning of the 3rd column of G?

(b) What is the meaning of the 2nd row of G?

(c) Give a simple formula for the n-vectors M, where M_i is the total membership in the group i.

(d) Interpret $(GG^T)_{ij}$ in simple Enlgish.

(e) Interpret $(G^T G)_{ij}$ in simple English.

Question 8

An n-vector x is symmetric if $x_k = x_{n-k+1}$ for $k = 1, \ldots, n.$ It is anti-symmetric if $x_k = -x_{n-k+1}$ for $k = 1, \ldots n.$

(a) Show that every vector x can be decomposed in a unique way as sum $x = x_s + x_a$ of a symmetric vector x_s and an anti-symmetric vector x_a.

(b) Show that the symmetric and anti-symmetric parts x_s and x_a are linear functions of x. Given matrices A_s and A_a such that $x_s = A_s x$, and $x_a = A_a x$ for all x.

Question 9

For descriptions of y below, express it as $y = Ax$ for some A (You should specify A).

(a) y_i is the difference between x_i and the average of $x_1, \ldots, x_{i-1}$. (We take $y_1 = x_1$).

(b) y_i is the difference between x_i and the average value of all other x_j. i.ie the average of $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n$

Question 10

Write your own code to implement the Grah-Schmidt procedure. Test your code and find an orthogonal basis of

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Question 11

The following code geneartes a signal together with noise.

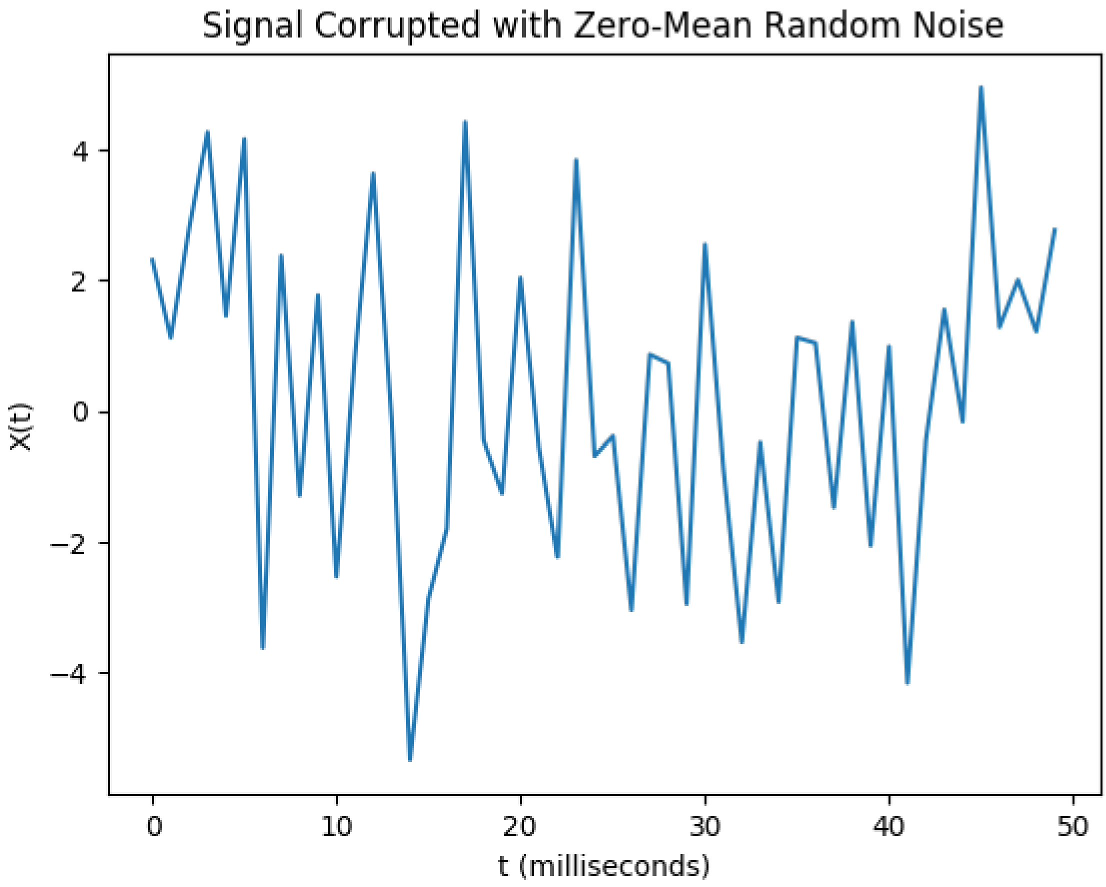
(a) Compute and plot the fourier transform of the signal without noise.

(b) Compute and plot the the fourier transform with signal noise.

(c) Corruput the signal with different white-noise, and reproduce the plot of (a) and (b).

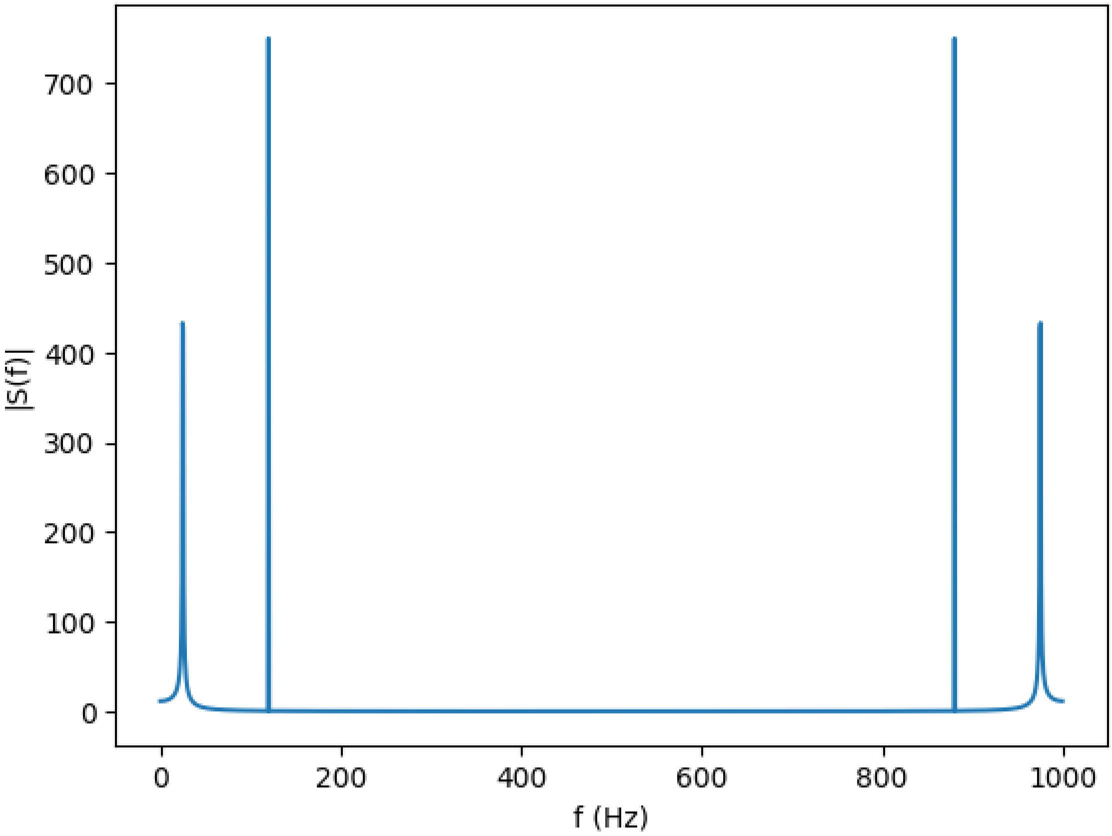
In [1]:

```
using PyPlot

Fs = 1000;              #Sampling frequency
T = 1/Fs;               #Sampling period
L = 1500;               #Length of signal
t = (0:L-1)*T;          #Time vector

#Form a signal containing a 25 Hz sinusoid of amplitude 0.9 and a 120 Hz sinusoi
d of amplitude 1.
S = 0.9*sin.(2*pi*25*t) + sin.(2*pi*120*t);
#Corrupt the signal with zero-mean white noise with a variance of 4.
X = S+2*randn(size(t));

plot(1000*t[1:50],X[1:50])
title("Signal Corrupted with Zero-Mean Random Noise")
xlabel("t (milliseconds)")
ylabel("X(t)");
```


Signal Corrupted with Zero-Mean Random Noise
X(t)
4
2
0
−2
−4
0
10
20
30
40
50
t (milliseconds)

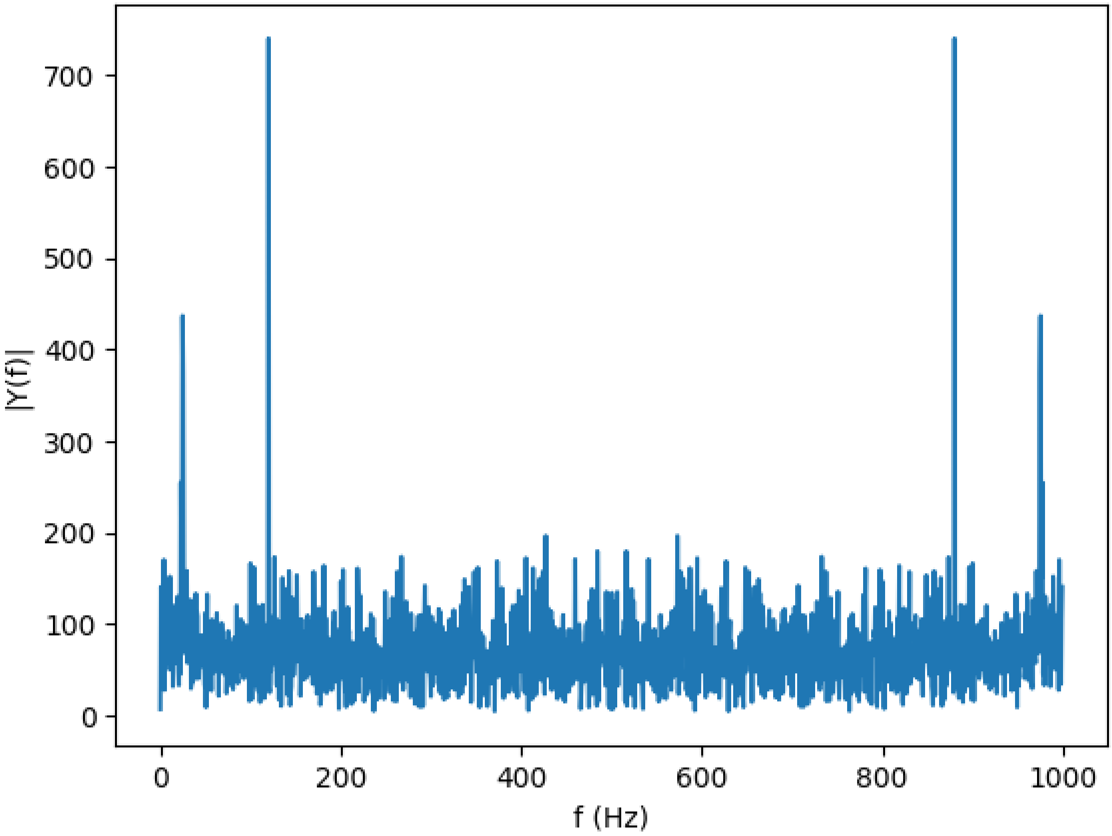
In [2]:

```
## Compute the FFT for the term without noise
Z=abs(fft(S))
## frequency axis
f = Fs*(0:(L-1))/L;
plot(f,Z)
xlabel("f (Hz)")
ylabel("|S(f)|");
```



In [3]:

```
## Compute the FFT for the term without noise
Y=abs(fft(X))
## frequency axis
f = Fs*(0:(L-1))/L;
plot(f,Y)
xlabel("f (Hz)")
ylabel("|Y(f)|");
```



Question 12

Explain the results of the previous question. How can Fourier analysis be used in data science? Present a 1-2 paragraph subjective description.