Assignment 3

MATH 7502 - Semsester 2, 2018

Mathematics for Data Science 1

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Question 1

Consider the following recurision,

$$x_{t+1} = A_1 x_t + A_2 x_{t-1}, \quad , t = 2, 3, \ldots$$

where x_t is n-vector and A_1 and A_2 are $n \times n$ matrices. Define $z_t = (x_t, x_{t-1})$.

Show that z_t satisfies the linear dynamical system equation $z_{t+1} = Bz_t$, for $t = 2, 3, \ldots,$ where B is a $(2n) \times (2n)$ matrix.

Question 2

Consider the Fibonacci sequence $y_0, y_1, y_2, \ldots$ with $y_0 = 0, y_1 = 1, y_2 = 1, y_3 = 2, \ldots,$ and for $t = 2, 3, \ldots, y_t$ is the sum of the previous two terms y_{t-1} and y_{t-2}.

(a) Express the Fibonacci sequence as a time-invaraint dynamical system wit state $x_t = (y_t, y_{t-1})$ and output y_t for $t = 1, 2, 3....$ as

$$x_{t+1} = Ax_t$$

(b)For the matrix, A, compute the eigenvalues and describe the Fibonnaci sequence interms of eigevnalues and eigenvectors. How does the golden ratio play a role?

Question 3

In the sepcial case $n = 1$, the general least square problem reduces to finding a scalar x that minimizes $\|ax - b\|^2$, where a and b are m-vectors. Assuming a and b are nonzero, show that $\|a\hat{x} - b\|^2 = \|b\|^2 \sin^2(\theta)$, where $\theta = \angle(a, b)$.

Question 4

Consider a time-invaraint linear dynamical system with offset

$$x_{t+1} = Ax_t + c$$

where x_t is the state n-vector. We say that a vector z is an equilibrium point of the linear dynamical system if $x_1 = z$ implies $x_2 = z, x_3 = z, \ldots$

(a) Find a matrix F and a vector g for which the set of linear equations $Fz = g$ characterizes equilibrium points. (This means: If z is an equilibrium point, then $Fz = g$; conversely if $Fz = g$, then z is an equilibrium point.)

Express F and g interms of A, c any standard matrices or vectors and matrix and vector operations.

Question 5

Suppose that $m \times n$ matrix Q has orthonormal columns and b is an m-vector. Show that $\hat{x} = Q^T b$ is the vector that minimizes $\|Qx - b\|^2$.

Comment on the complexity of finding $\hat{x}$ given Q and b in this case. Compare the complexity with the general leasure square problem where Q is a coefficient matrix.

Question 6

Suppose $m \times n$ matrix A has linearly independent columns, and b is a m-vector. Let $\hat{x} = A^\dagger b$ denote the least squares approximate solution of $Ax = b$.

(a) Show that for any n-vector x, $(Ax)^T b = (Ax)^T (A\hat{x})$. Hint: Use $(Ax)^T b = x^T (A^T b)$, and $(A^T A)\hat{x} = A^T b$.

(b) Show that when $A\hat{x}$ and b are both nonzero, we have

$$\frac{(A\hat{x})^T b}{\|A\hat{x}\| \|b\|} = \frac{\|A\hat{x}\|}{\|b\|}$$

.

(c) The choice of $x = \hat{x}$ minimizes the distance between Ax and b, Show that $x = \hat{x}$ also minimizes the angle between Ax and b.

Question 7

Suppose A is an $m \times n$ matrix with linearly independent columns and QR factorization $A = QR$, and b is the m-vector. The vector $A\hat{x}$ is the linear combination of the columns of A that is closet to the vector b, i.e., it is the projection of b onto the set of linear combinations of the columns of A.

(a) Show that $A\hat{x} = QQ^T b$.

(b) Show that $\|A\hat{x} - b\|^2 = \|b\|^2 - \|Q^T b\|^2$.

Question 8

A generalization of the least squares problem adds an affine function to the least squares objective

$$\text{minimize} \qquad \|Ax - b\|^2 + c^T x + d$$

where x is an n-vector as a variable to be chosen, and the data are the $m \times n$ matrix A, the m-vector b, the n-vector c. and the number d. The columns of A are linearly indeppendent.

Show that that objective of the problem above can be expressed in the form

$$\|Ax - b\|^2 + c^T x + d = \|Ax - b + f\|^2 + g$$

,

for some m-vector f and some constant g. It follows that we can solve the generalized least squares problem by minimizing $\|Ax - (b - f)\|$, an ordinary least squares problem with solution $\hat{x} = A^\dagger(b - f)$.

Hint: Express the norm squared term on the right-hand side as $\|(Ax - b) + f\|^2$ and expand it.

Question 9

A very simple model of how the economic output changes over time is $a_{t+1} = Ba_t$, where B is an $n \times n$ matrix, $(a_t)_i$ is the economic output in sector i in year t. In this problem, we will consider the specific model with $n = 4$ and

$$B = \begin{bmatrix} 0.1 & 0.06 & 0.05 & 0.7 \\ 0.48 & 0.44 & 0.10 & 0.04 \\ 0.00 & 0.55 & 0.52 & 0.04 \\ 0.04 & 0.01 & 0.42 & 0.51 \end{bmatrix}$$

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(a) Breifly interpret B_{23}.

(b) Suppose $a_1 = (0.6, 0.9, 1.3, 0.05)$ plot four sector outputs and the total economic output versus t for $t = 1, \ldots, 20$.

Question 10

Solve Question 12.13 from [VMLS], page 241.

Question 11

You are given a channel impulse response, the n-vector c. Your job is to find an equalizer impulse response, the n-vector h that minimizes $\|h * c - e_1\|^2$. You can assume $c_1 \neq 0$.

(a) Explain how to find h, Apply your method to find the equalizer h for the channel $c = (1.0, 0.7, -0.3, -0.1, 0.05)$.

(b) Plot c, h, and $h * c$.

Question 12

Consider the linear dynamical system,

$$\dot{x}_t = Ax_t.$$

with $x_0 = x_0$ and A an $n \times n$ matrix.

Plot trajectories of such a system with $n = 2$ and $x_0 = (1, 1)'$ for the following numerical example cases:

- Both eigenvalues of A real and negative.
- Both eigenvalues of A real and positive.
- One eigenvalue real positive and one real negative.
- Both eigenvalues complex and negative.
- Pure imaginary eigenvalues.