Assignment 5

MATH 7502 - Semsester 2, 2018

Mathematics for Data Science 2

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Question 1

Suppose G_{k+2} is the average of two previous numbers G_{k+1} and G_k

$$G_{k+2} = \frac{1}{2}(G_{k+1} + G_k), \quad G_{k+1} = G_{k+1}.$$

(a) Find the eigenvalues and eigenvectors of matrix A such that

$$\begin{bmatrix} G_{k+2} \\ G_{k+1} \end{bmatrix} = A \begin{bmatrix} G_{k+1} \\ G_k \end{bmatrix}$$

(b) Find the limit as $n \to \infty$ of the matrices $A^n = X\Lambda X^{-1}$.

(c) If $G_0 = 0, G_1 = 1$, find the limit.

Question 2

Let $A = \begin{bmatrix} 6 & 3 & -2 \\ -4 & -1 & 2 \\ 13 & 9 & -3 \end{bmatrix}$.

(a) Evaluate e^{tA}.

(b) Find the general solutions of $\frac{d\vec{x}}{dt} = A\vec{x}$.

(c) Solve the initial value problem, if $\overrightarrow{x(0)} = \begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix}$.

(d) Give an example of matrix M and N such that

$$e^M e^N \neq e^{M+N}$$

Question 3

Find the singular value decomposition $U\Sigma V^T$ of A, where

$$A = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix}.$$

(a)What is the rank of A?

(b) Suggest two rank 1 approximations of A based on the SVD. Which one is better?

(c) Write code that shows evidence of (b).

Question 4

Suppose we take a sample of 7 from the students' tests in Math, Science and History, the result is the following:

Math	Science	History
7	5	4
2	1.5	2.0
3	9	8
3	7	7
5	3.5	5.0
6	4.5	5.0
7	3.5	4.5

(a) Compute the sample covariance matrix S.

(b) Find the eigenvalues of S.

(c) Write code for (a) and (b).

(d) Comment on the principal componenent analysis aspects of this and how they can be used.

Question 5

Suppose $Ax = \lambda x$. If $\lambda = 0$ then x is in the nullspace. If $\lambda \neq 0$ then x is in the column space. Those spaces have dimensions $(n - r) + r = n$. Show that not every square matrix has n linearly independent eigenvectors?

Question 6

Consider a data-set with $2M$ points, each a 2-vector.

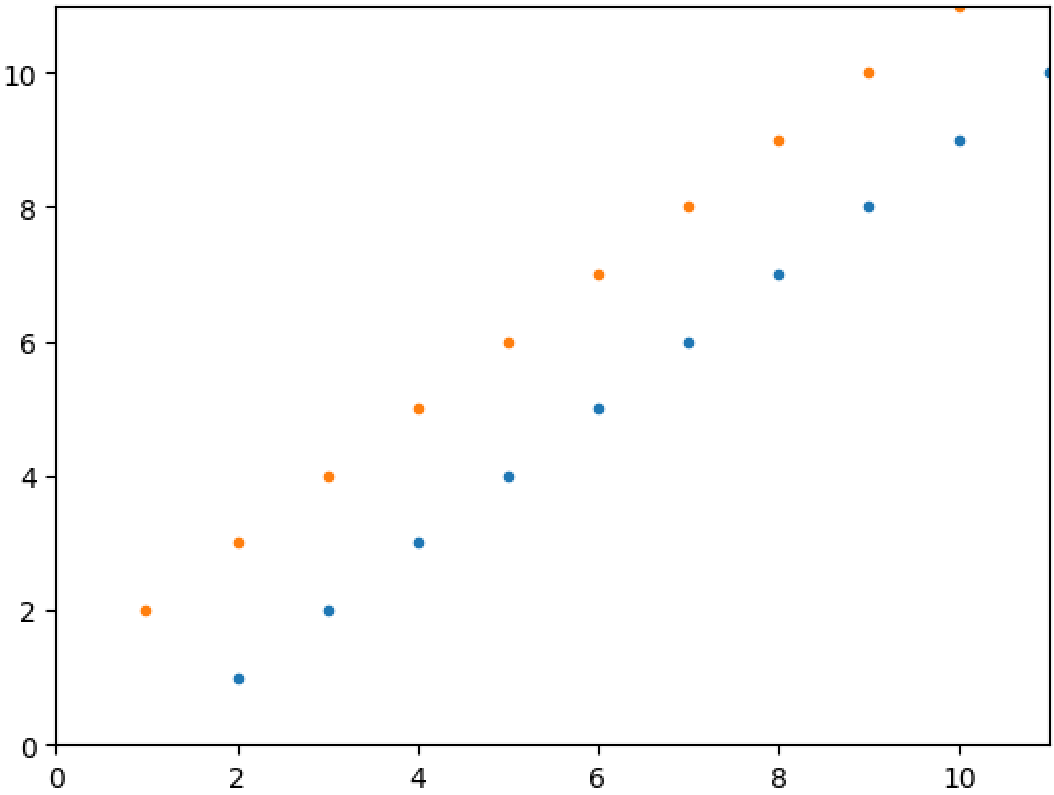
The first M points are given by, $i = 1, \dots, M$ via the vectors $(i + 1, i)$.

The later M points are given by, $i = 1, \dots, M$ via the vectors $(i, i + 1)$.

For example, for $M = 10$ a plot of the points is given below.

In [62]:

```
using PyPlot
M = 10
points1 = [[i+1,i] for i in 1:M]
points2= [[i,i+1] for i in 1:M]
plot(first.(points1),last.(points1),".")
plot(first.(points2),last.(points2),".")
xlim(0,M+1);ylim(0,M+1);
```



(a) Give an explicit expression for the normalized covariance matrix of these points. As an aid, below is a numerical computation of the covariance matrix.

In [63]:

```
X = [reshape(vcat(points1...),2,M) reshape(vcat(points2...),2,M)]
muX = X*ones(2M,2M)/2M
cvMat = (X-muX)*(X-muX)'/(2M)
```

Out[63]:

```
2×2 Array{Float64,2}:
 8.5  8.0
 8.0  8.5
```

(b) Carry out PCA for this data-set, numerically.

(c) Carry out PCA analytically and compare to your numerical results

Question 7

Each pair of singular vectors v and u has $Av = \sigma u$ and $A^T u = \sigma v$.

(a) Show that the double vector $\begin{bmatrix} u \\ v \end{bmatrix}$ is an eigenvector of the symmetric block matrix $M = \begin{bmatrix} 0 & A^T \\ A & 0 \end{bmatrix}$.

(b) Show that the SVD of A is equivalent to the diagonalization of M.

Question 8

Suppose the same X diagonalizes both A and B. They have same eigenvectors in $A = X\Lambda_1 X^{-1}$ and $B = X\Lambda_2 X^{-1}$. Prove that $AB = BA$.

Question 9

Consider the code below.

(a) Describe in words what the code does.

(b) Replace line 5 of the code with a different implemination of pixel(). Have it create a square-looking donut. Carry out the computation with this image. Comment qualitatively on your results.

In [ ]:

```
using PyPlot,Distributions

n,m = 100,100
sig = 0.1
pixel(i,j) = ( (i-50)^2+(j-50)^2 < 25^2 && (i-50)^2+(j-50)^2 > 10^2) +
sig*rand(Normal())
A = [pixel(i,j) for i in 1:n, j in 1:m]

p = min(n,m)
U,S,V = svd(A);
svdApprox(k) = U[:,1:k]*diagm(S[1:k])*V[:,1:k]'
err = [norm(A-svdApprox(k)) for k in 1:p]
plot(1:p,err)
```

In [ ]:

```
fig = figure(figsize=(10,10))
d = 15
for k in 1:d
    imm = fig[:add_subplot](d,1,k)
    imm[:imshow](svdApprox(k),cmap="Greys")
end
```

Question 10

This is a carry over from Question 10 of Assignment 4. If you didn't do it there, do it now: Carry out 16.5 from [VMLS], page 352.

Question 11

This is a carry over from Question 11 of Assignment 4. If you didn't do it there, do it now

Question 12

This is a carry over from Question 12 of Assignment 4. If you didn't do it there, do it now