Least Squares for Data Science

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Created using Julia 1.1

In [1]:
```
using LinearAlgebra, Statistics, Plots, Random, Distributions
pyplot();
Random.seed!(0);
```

In [2]:
```
n = 10;
m = 3
A = rand(1:5,n,m)
```

Out[2]:
```
10×3 Array{Int64,2}:
 1  3  4
 3  5  1
 2  2  1
 1  3  3
 3  3  5
 1  2  5
 4  1  3
 1  5  4
 4  4  4
 4  5  3
```

In [3]:
```
b = rand(100:20:200,n)
```

Out[3]:
```
10-element Array{Int64,1}:
 200
 160
 140
 180
 180
 120
 120
 200
 120
 140
```

We'll need to be very lucky to have an exact solution, x, for

$$Ax = b.$$

It would mean that b is in the collumn space (range) of A.

In [4]:
```
bt = 0.3A[:,1] - 5A[:,2] + 4A[:,3]
```

Out[4]:
```
10-element Array{Float64,1}:
   1.3000000000000007
 -20.1
  -5.4
  -2.6999999999999993
   5.9
  10.3
   8.2
  -8.7
  -2.8000000000000007
 -11.8
```

In [5]:
```
x = inv(A[1:3,:])*bt[1:3]
```

Out[5]:
```
3-element Array{Float64,1}:
  0.3000000000000016
 -5.000000000000003
  4.000000000000001
```

In [6]:
```
A*x - bt
```

Out[6]:
```
10-element Array{Float64,1}:
 -3.552713678800501e-15
 -7.105427357601002e-15
 -8.881784197001252e-16
 -3.552713678800501e-15
  1.7763568394002505e-15
  0.0
  7.105427357601002e-15
 -1.0658141036401503e-14
  0.0
 -3.552713678800501e-15
```

However in general we are typically not so lucky...

In [7]:
```
x = inv(A[1:3,:])*b[1:3]
A*x - b
```

Out[7]:
```
10-element Array{Float64,1}:
  -5.684341886080802e-14
  -1.4210854715202004e-13
  -5.684341886080802e-14
 -25.000000000000057
 187.49999999999994
 138.75
 246.25
 -27.500000000000114
 249.9999999999999
 171.2499999999999
```

So instead we try to minimize $||Ax - b||$ or similarly $||Ax - b||^2$.

In one line this is how we get the least squares minimizer:

In [8]:
```
1 xHat = A\b
```

Out[8]:
```
3-element Array{Float64,1}:
 -0.7213250090294512
 25.268046024456932
 20.087714772199572
```

In [9]:
```
1 A*xHat - b
```

Out[9]:
```
10-element Array{Float64,1}:
 -44.56632784686036
 -15.736030132604128
 -70.81884319694547
 -44.654042619059965
  -5.921263092719698
  30.25334090088228
 -37.354109695062164
   5.969764202053511
  58.53774315050822
  43.71807440276558
```

In [10]:
```
1 bestLoss = norm(A*xHat - b)
```

Out[10]:
```
130.23940757974952
```

Let's try values "around it" to get convinced about its optimallity...

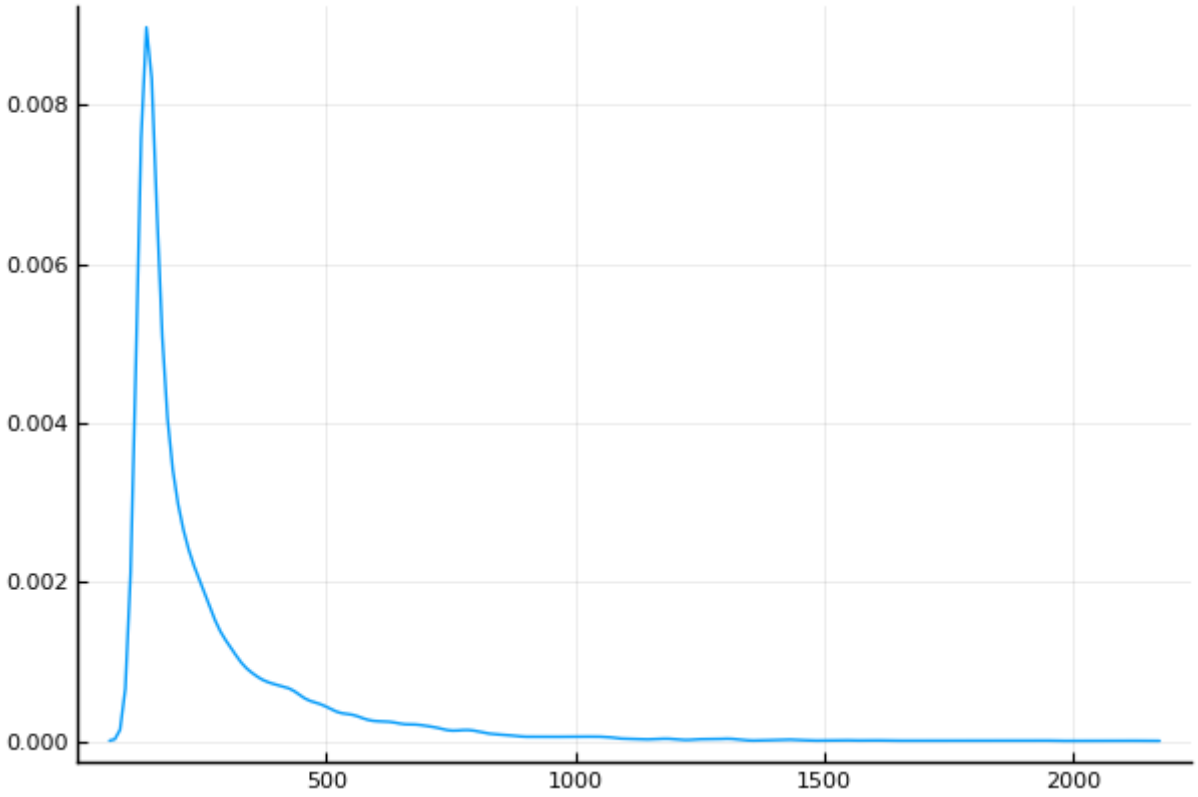
In [11]:

```
loss(x) = norm(A*x - b)

R1(θ) = [cos(θ) -sin(θ) 0;
         sin(θ) cos(θ) 0 ;
           0      0    1 ]

R2(θ) = [cos(θ)  0 -sin(θ);
           0      0    1 ;
         sin(θ)  0 cos(θ) ]


pts = []

N = 10000

for _ in 1:N
    θ1,θ2 = rand(Uniform(0,2π),2)
    r = 20*randexp()
    xt = xHat + R1(θ1)*R2(θ2)*[r,0,0]
    push!(pts,[xt ; loss(xt)])
    if loss(xt) < loss(xHat)
        println("Found something better")
    end
end
losses = last.(pts)
println("Minimum explored loss: ", minimum(losses) ,"\tvs. optimum loss:
```

```
Minimum explored loss: 130.23941151184172       vs. optimum loss: 130.239
40757974952
```

In [12]:
```
using StatsPlots
density(losses,legend=false)
```

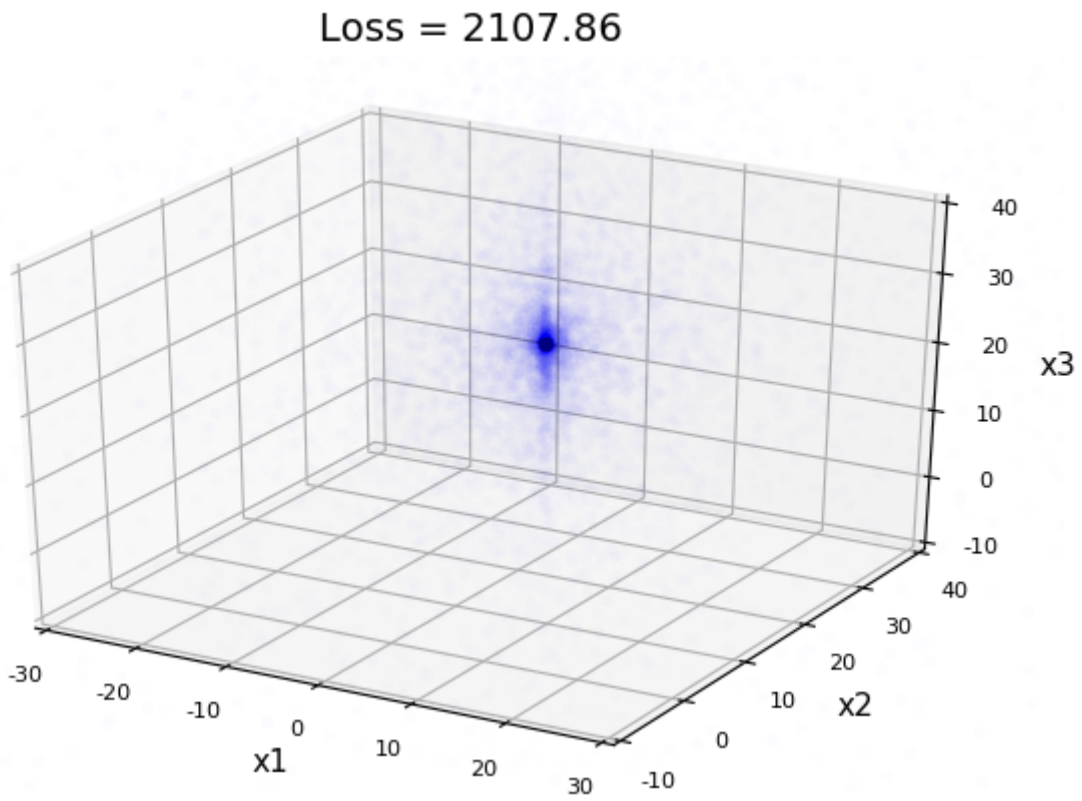
Out[12]:



We can even make a little movie to try and visualize such a minimum point...

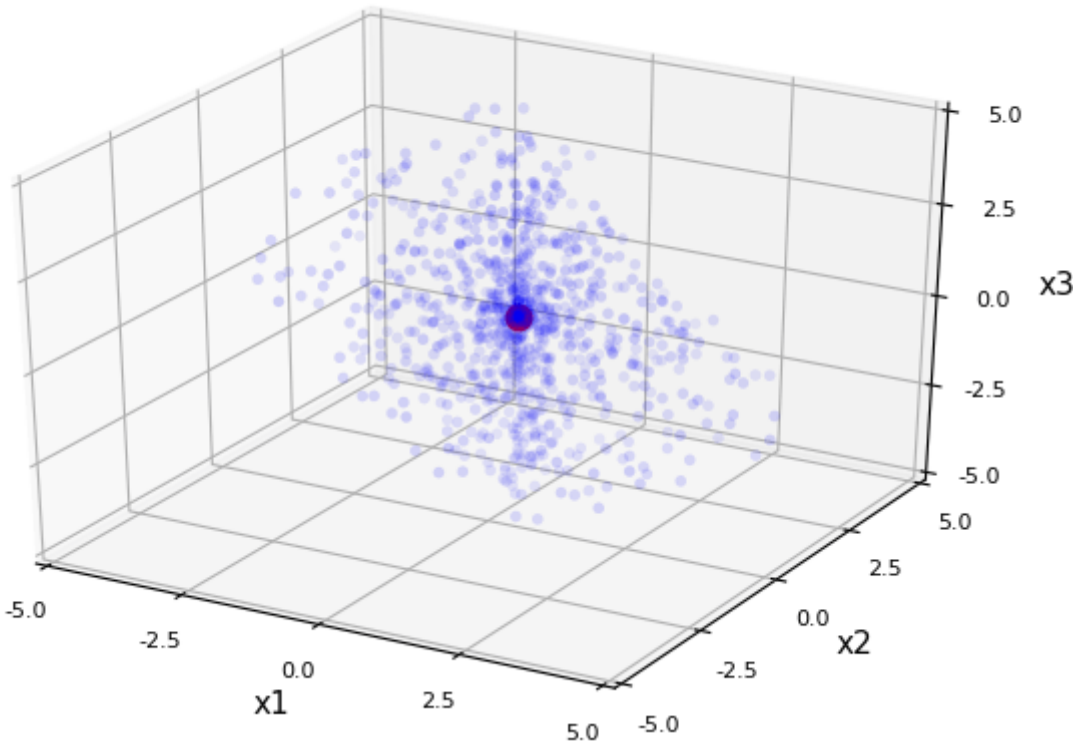
In [13]:

```
ani = Animation()
for α in 0.01:0.01:1.0
    lossQ = quantile(losses,α)
    pltPnts = filter(x->x[4] < lossQ,pts)
    scatter([xHat[1]],[xHat[2]],[xHat[3]],c=:red,ms=5.0,markerstrokewidt
    scatter!(first.(pltPnts),(x->x[2]).(pltPnts),(x->x[3]).(pltPnts),
        c=:blue,markeralpha=0.01,markerstrokewidth=0,xlabel="x1",ylabel=
        xlim=(-30,30),ylim=(-10,40),zlim=(-10,40),title="Loss = $(round(
    frame(ani)
end
gif(ani,"bestValue.gif",fps=10)
```




```
┌ Info: Saved animation to
|   fn = /Users/yoninazarathy/Dropbox/MATH7502/LeastSquaresVideo/bestValu
e.gif
└ @ Plots /Users/yoninazarathy/.julia/packages/Plots/Ih71u/src/animation.
jl:95
```

Out[13]:



Getting there via gradient descent

$$L(x) = \|Ax - b\|^2 = (Ax - b)^T(Ax - b)$$

$$\nabla L(x) = 2A^T(Ax - b).$$

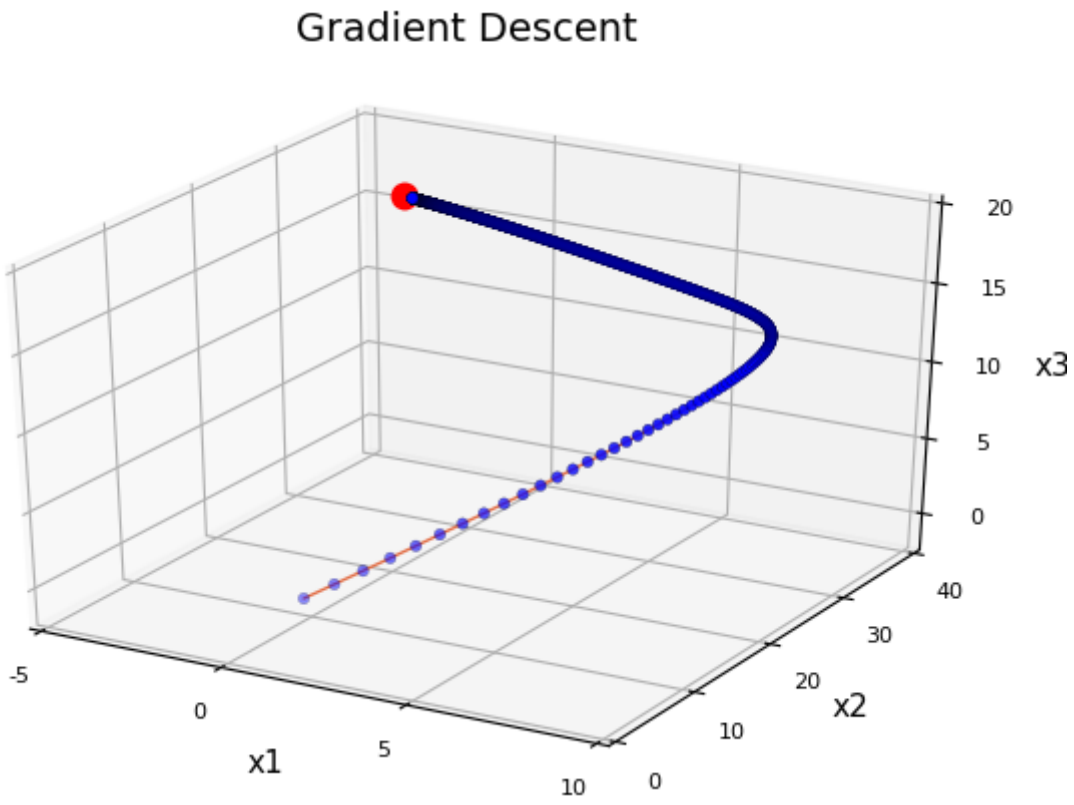
In [14]:

```
traj = []
x = ones(3)
xprev = -x
η = 0.0001
while norm(x-xprev) > 10^-3
    xprev = x
    x = x - η*2A'*(A*x-b)
    push!(traj,x)
end
```

In [15]:

```
scatter(first.(traj),(x->x[2]).(traj),(x->x[3]).(traj),
        c=:blue,markeralpha=1,markerstrokewidth=0.2,xlabel="x1",ylabel="
        xlim=(-5,10),ylim=(0,40),zlim=(-2,20),title="Gradient Descent")
plot!(first.(traj),(x->x[2]).(traj),(x->x[3]).(traj))
scatter!([xHat[1]],[xHat[2]],[xHat[3]],c=:red,ms=10.0,markerstrokewidth=
```

Out[15]:



The Normal Equations and $A^\dagger$

$$\nabla L(x) = 2A^T(Ax - b) = 0$$

$$2A^TAx = 2A^Tb$$

$$x = (A^TA)^{-1}A^Tb = A^\dagger b$$

In [16]:
```
1 xHat
```

Out[16]:
```
3-element Array{Float64,1}:
 -0.7213250090294512
 25.268046024456932
 20.087714772199572
```

In [17]:
```
1 pinv(A)
```

Out[17]:
```
3×10 Array{Float64,2}:
 -0.0590785   0.0235282   0.0419483  …  -0.103555    0.0635674    0.05149
37
  0.0171611   0.101986    0.0120943      0.0919148  -0.0121769    0.04338
27
  0.0523915  -0.0921315  -0.0274186      0.016026    0.00371498  -0.04374
39
```

In [18]:
```
1 inv(A'*A)*A'
```

Out[18]:
```
3×10 Array{Float64,2}:
 -0.0590785   0.0235282   0.0419483  …  -0.103555    0.0635674    0.05149
37
  0.0171611   0.101986    0.0120943      0.0919148  -0.0121769    0.04338
27
  0.0523915  -0.0921315  -0.0274186      0.016026    0.00371498  -0.04374
39
```

In [19]:
```
1 pinv(A)*b
```

Out[19]:
```
3-element Array{Float64,1}:
 -0.7213250090294663
 25.26804602445693
 20.087714772199597
```

In [20]:
```
1 A'A \ A'b
```

Out[20]:
```
3-element Array{Float64,1}:
 -0.7213250090294668
 25.268046024456947
 20.087714772199575
```

Using Factorizations for $A^\dagger$

The QR factorization: $A = QR$

$$A^\dagger = (A^TA)^{-1}A^T = (R^TQ^TQR)^{-1}R^TQ^T = (R^TR)^{-1}R^TQ^T = R^{-1}Q^T$$

So a least squares approximate solution can be obtained via $\hat{x} = R^{-1}Q^Tb$

In [21]:
```
F = qr(A);
Q, R = F.Q, F.R
R
```

Out[21]:
```
3×3 Array{Float64,2}:
 -8.60233  -9.41606  -8.83482
  0.0      -6.19176  -3.84557
  0.0       0.0       5.84445
```

In [22]:
```
Q'*Q
```

Out[22]:
```
10×10 Array{Float64,2}:
  1.0           1.38778e-16  -6.93889e-17  …  1.66533e-16   1.11022e-16
  1.38778e-16   1.0          -1.63064e-16     5.89806e-17  -5.55112e-17
 -6.93889e-17  -1.63064e-16   1.0             2.77556e-17   8.32667e-17
  1.12323e-16  -1.95156e-18  -9.84456e-17     3.79471e-17   1.30104e-17
  2.63678e-16   7.80626e-17  -1.249e-16       7.28584e-17   1.38778e-17
  1.50921e-16   9.45424e-17  -2.48933e-16  …  5.20417e-18  -4.51028e-17
  2.35922e-16   2.498e-16     1.11022e-16     2.08167e-17   2.77556e-17
  1.52656e-16  -9.1073e-17   -1.79544e-16     5.55112e-17  -2.42861e-17
  1.66533e-16   5.89806e-17   2.77556e-17     1.0           2.77556e-17
  1.11022e-16  -5.55112e-17   8.32667e-17     2.77556e-17   1.0
```

In [23]:
```
inv(R)*Q'*b
```

Out[23]:
```
3-element Array{Float64,1}:
 -0.7213250090294547
 25.268046024456922
 20.087714772199593
```

The SVD factorization: $A = U\Sigma V^T$

$$A^\dagger = (A^TA)^{-1}A^T = (V\Sigma U^T U\Sigma V^T)^{-1}V\Sigma U^T = (V\Sigma^2 V^T)^{-1}V\Sigma U^T = V^{-T}\Sigma^{-2}V^{-T}V\Sigma U^T = V\Sigma$$

So a least squares approximate solution can be obtained via $\hat{x} = V\Sigma^{-1}U^T$

In [24]:
```
F = svd(A)
U, S, V = F.U , F.S, F.V;
S
```

Out[24]:
```
3-element Array{Float64,1}:
 17.05423938497788
  4.693241515159658
  3.889267679171903
```

In [25]:
```
sqrt.(eigvals(A'*A))
```

Out[25]:
```
3-element Array{Float64,1}:
  3.8892676791719025
  4.6932415151596585
 17.05423938497788
```

In [26]:
```
1 eigvals(A*A')
```

Out[26]:
```
10-element Array{Float64,1}:
 -9.30326999185546e-15
 -4.248162038514072e-15
 -1.33054744510433e-15
 -6.937870285002419e-16
  3.6308982445650267e-16
  4.900403029047245e-15
  6.366773908623675e-14
 15.126403080251208
 22.02651591961812
290.84708100013074
```

In [27]:
```
1 sqrt.(filter(x->x>0,eigvals(A'*A)))
```

Out[27]:
```
3-element Array{Float64,1}:
  3.8892676791719025
  4.6932415151596585
 17.05423938497788
```

In [28]:
```
1 A*V[:,1] - S[1]*U[:,1]
```

Out[28]:
```
10-element Array{Float64,1}:
 7.993605777301127e-15
 1.7763568394002505e-15
 8.881784197001252e-16
 8.881784197001252e-16
 2.6645352591003757e-15
 8.881784197001252e-16
 2.6645352591003757e-15
 8.881784197001252e-16
 1.7763568394002505e-15
 8.881784197001252e-16
```

In [29]:
```
1 pinv(A)
```

Out[29]:
```
3×10 Array{Float64,2}:
 -0.0590785   0.0235282   0.0419483  …  -0.103555    0.0635674    0.05149
37
  0.0171611   0.101986    0.0120943      0.0919148  -0.0121769    0.04338
27
  0.0523915  -0.0921315  -0.0274186      0.016026    0.00371498  -0.04374
39
```

In [30]:
```
V*Diagonal(1 ./ S)*U'
```

Out[30]:
```
3×10 Array{Float64,2}:
 -0.0590785   0.0235282   0.0419483  …  -0.103555    0.0635674    0.05149
37
  0.0171611   0.101986    0.0120943      0.0919148  -0.0121769    0.04338
27
  0.0523915  -0.0921315  -0.0274186      0.016026    0.00371498  -0.04374
39
```

What if $(A^T A)$ is singular? SVD to the rescue...

In [31]:
```
s = 0
while true
    Random.seed!(s)
    A = rand([1,2],n,m)
    r = rank(A)
    if rank(A) < 3
        println("Found rank rank $(r) matrix with seed s=$s")
        break
    end
    s += 1
end
```

```
Found rank rank 2 matrix with seed s=204
```

In [32]:
```
Random.seed!(204);
A = rand([1.0,2.0],n,m)
```

Out[32]:
```
10×3 Array{Float64,2}:
 2.0  1.0  2.0
 1.0  2.0  1.0
 1.0  1.0  1.0
 1.0  2.0  1.0
 1.0  1.0  1.0
 2.0  2.0  2.0
 1.0  1.0  1.0
 1.0  1.0  1.0
 2.0  2.0  2.0
 1.0  2.0  1.0
```

In [33]:
```
rank(A)
```

Out[33]:
```
2
```

In [34]:
```
rank(A'*A),det(A'*A)
```

Out[34]:
```
(2, 0.0)
```

In [35]:
```
1 inv(A'*A)
```

```
SingularException(3)

Stacktrace:
 [1] checknonsingular at /Users/julia/buildbot/worker/package_macos64/bui
ld/usr/share/julia/stdlib/v1.1/LinearAlgebra/src/factorization.jl:12 [inl
ined]
 [2] #lu!#103(::Bool, ::Function, ::Array{Float64,2}, ::Val{true}) at /Us
ers/julia/buildbot/worker/package_macos64/build/usr/share/julia/stdlib/v
1.1/LinearAlgebra/src/lu.jl:41
 [3] #lu! at ./none:0 [inlined]
 [4] #lu#107 at /Users/julia/buildbot/worker/package_macos64/build/usr/sh
are/julia/stdlib/v1.1/LinearAlgebra/src/lu.jl:142 [inlined]
 [5] lu at /Users/julia/buildbot/worker/package_macos64/build/usr/share/j
ulia/stdlib/v1.1/LinearAlgebra/src/lu.jl:142 [inlined] (repeats 2 times)
 [6] inv(::Array{Float64,2}) at /Users/julia/buildbot/worker/package_maco
s64/build/usr/share/julia/stdlib/v1.1/LinearAlgebra/src/dense.jl:732
 [7] top-level scope at In[35]:1
```

So it appears that finding a least squares approximate solution to $Ax = b$ isn't doable...???

In [36]:
```
1 A \ b
```

Out[36]:
```
3-element Array{Float64,1}:
 29.333333333333293
 44.266666666666694
 29.333333333333307
```

In [37]:
```
1 aDag = pinv(A)
```

Out[37]:
```
3×10 Array{Float64,2}:
  0.2   -0.1    0.0333333  -0.1   …   0.0333333   0.0666667  -0.1
 -0.28   0.24  -0.0133333   0.24     -0.0133333  -0.0266667   0.24
  0.2   -0.1    0.0333333  -0.1       0.0333333   0.0666667  -0.1
```

In [38]:
```
1 aDag*b
```

Out[38]:
```
3-element Array{Float64,1}:
 29.333333333333346
 44.266666666666644
 29.333333333333343
```

In [39]:
```
1 F = svd(A)
2 U, S, V = F.U , F.S, F.V;
3 S
```

Out[39]:
```
3-element Array{Float64,1}:
 7.779559352679487
 1.5743113663240509
 4.741575244451873e-16
```

In [40]:
```
S = filter(x->x>10^-6,S)
```

Out[40]:
```
2-element Array{Float64,1}:
 7.779559352679487
 1.5743113663240509
```

In [41]:
```
r = length(S)
```

Out[41]:
```
2
```

In [42]:
```
V[:,1:r]*Diagonal(1 ./ S)*U[:,1:r]'
```

Out[42]:
```
3×10 Array{Float64,2}:
  0.2    -0.1     0.0333333  -0.1    …   0.0333333   0.0666667  -0.1
 -0.28    0.24   -0.0133333   0.24      -0.0133333  -0.0266667   0.24
  0.2    -0.1     0.0333333  -0.1        0.0333333   0.0666667  -0.1
```

In [43]:
```
pinv(A)
```

Out[43]:
```
3×10 Array{Float64,2}:
  0.2    -0.1     0.0333333  -0.1    …   0.0333333   0.0666667  -0.1
 -0.28    0.24   -0.0133333   0.24      -0.0133333  -0.0266667   0.24
  0.2    -0.1     0.0333333  -0.1        0.0333333   0.0666667  -0.1
```

What is going on? Back to the normal equations...

$$A^T A x = A^T b$$

And using $x = V\Sigma^{-1} U^T$

In [44]:
```
A'*A*(pinv(A)*b) - A'*b
```

Out[44]:
```
3-element Array{Float64,1}:
 0.0
 0.0
 0.0
```

$$A^T A x = A^T b$$

$$(U\Sigma V^T)^T U\Sigma V^T x = (U\Sigma V^T)^T b$$

$$V\Sigma^2 V^T x = V\Sigma U^T b$$

Now notice that $V\Sigma^{-2} V^T$ is a left inverse of $V\Sigma^2 V^T$. So left multiply by it:

$$x = V\Sigma^{-2} V^T V\Sigma U^T b$$

Hence,

$$x = V\Sigma^{-1} U^T b$$

So in summary, we see that using the SVD for the pseudo-inverse yields a robust solution. Note that

when A isn't full rank (and thus A^TA is singular), there are many solutions to the normal equations. It can be shown that the one based on the SVD based left-inverse is the minimal solution...

In [45]:
```
rank(A'*A)
```

Out[45]:
```
2
```

The next topic we explore, Tikhonov regularization (ridge regression) is related... Think that we modify A^TA by adding a term:

$$A^TA + \lambda I$$

with $\lambda > 0$

In [46]:
```
AA(λ) = A'*A + λ*I
inv(AA(0.01))*A'
```

Out[46]:
```
3×10 Array{Float64,2}:
  0.199296  -0.0995153   0.0332603  …   0.0332603   0.0665206  -0.0995153
 -0.278762   0.239129   -0.0132112      -0.0132112  -0.0264223   0.239129
  0.199296  -0.0995153   0.0332603      0.0332603   0.0665206  -0.0995153
```

In [47]:
```
pinv(A)
```

Out[47]:
```
3×10 Array{Float64,2}:
  0.2   -0.1    0.0333333  -0.1   …   0.0333333   0.0666667  -0.1
 -0.28   0.24  -0.0133333   0.24     -0.0133333  -0.0266667   0.24
  0.2   -0.1    0.0333333  -0.1       0.0333333   0.0666667  -0.1
```

In [48]:
```
1 [norm(pinv(A) - inv(AA(λ))*A') for λ in 0.5:-0.001:0.001]
```

Out[48]:
```
500-element Array{Float64,1}:
 0.10663734620705814
 0.1064598114844861
 0.10628215748348022
 0.10610438408377042
 0.10592649116492865
 0.10574847860636352
 0.10557034628731969
 0.10539209408688131
 0.10521372188396942
 0.10503522955734089
 0.10485661698558987
 0.10467788404714788
 0.10449903062028054
 ⋮
 0.0030607421523459345
 0.002806807256767531
 0.002552668284534555
 0.0022983249895064286
 0.002043777125163965
 0.0017890244445544544
 0.001534066700416769
 0.0012789036449568556
 0.0010235350300804545
 0.0007679606072593527
 0.0005121801275852495
 0.0002561933417423063
```

Regularization

The idea is to to add a weighted term of $||x||^2$. The objective is then:

$$L_\lambda(x) = ||Ax - b||^2 + \lambda||x||^2 = || \begin{bmatrix} A \\ \sqrt{\lambda} I \end{bmatrix} - \begin{bmatrix} b \\ 0 \end{bmatrix} ||^2$$

Here,

$$\tilde{A} = \begin{bmatrix} A \\ \sqrt{\lambda} I \end{bmatrix} \qquad \tilde{b} = \begin{bmatrix} b \\ 0 \end{bmatrix}$$

And hence $\hat{x} = (\tilde{A}^T \tilde{A})^{-1} \tilde{A}^T \tilde{b} = (A^T A + \lambda I)^{-1} A^T b$

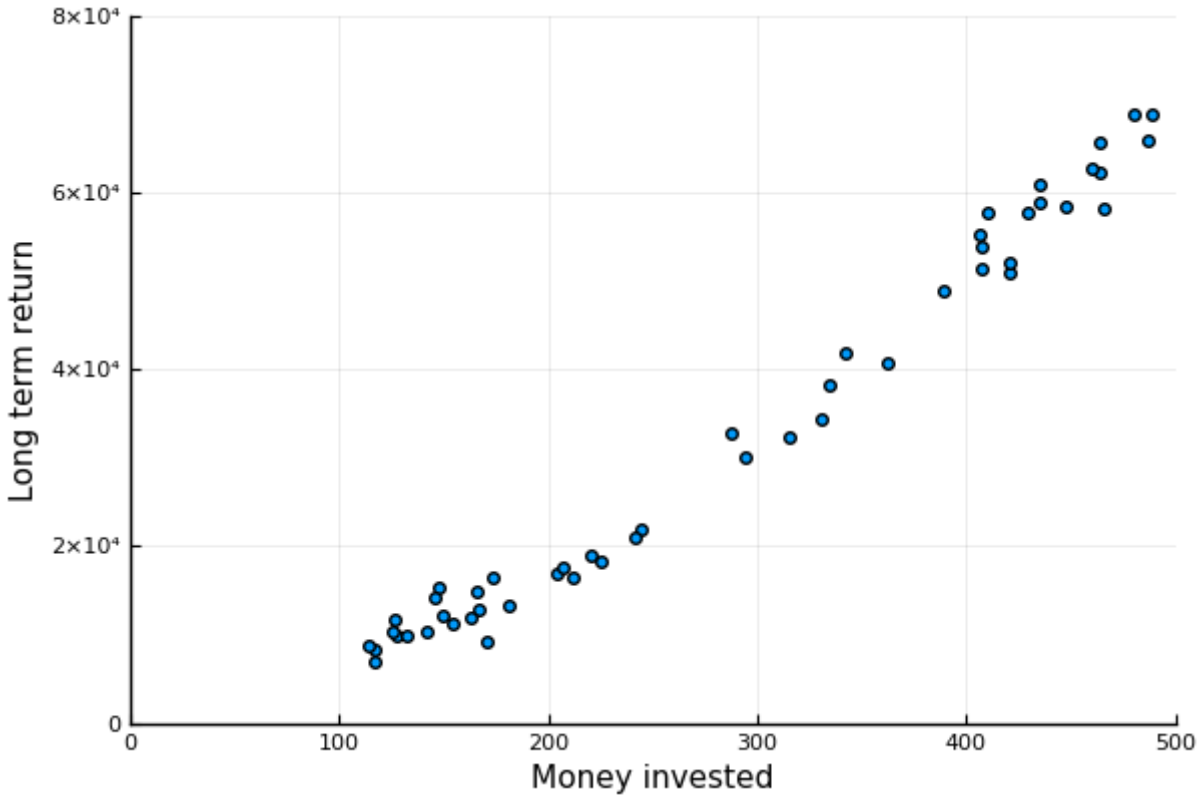
Data Fitting

Some reality...

In [49]:

```
Random.seed!(0)
reality(x) = 250 + 30x + 0.8x^1.8 + 2000randn()
n = 50;
xVals = 100 .+ 400rand(n)
yVals = reality.(xVals);
scatter(xVals,yVals,legend=false,
    xlabel="Money invested", ylabel = "Long term return",xlim=(0,500),yl
```

Out[49]:



Say we look at this data and decide to fit a curve of the form

$$y = \beta_0 + \beta_1 x + \beta_2 x^2 = \begin{bmatrix} 1 & x & x^2 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{bmatrix}$$

We can now set the **design matrix**:

$$A = \begin{bmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \\ \vdots & \vdots & \vdots \\ 1 & x_n & x_n^2 \end{bmatrix}$$

And seek a β vector that minimizes $\|A\beta - y\|$.

In [50]:
```
1 A =[xVals[i]^j for i in 1:n, j in 0:2]
```

Out[50]:
```
50×3 Array{Float64,2}:
 1.0  429.459  184435.0
 1.0  464.143       2.15428e5
 1.0  165.826   27498.4
 1.0  170.932   29217.6
 1.0  211.552   44754.3
 1.0  181.391   32902.6
 1.0  116.921   13670.4
 1.0  127.308   16207.3
 1.0  244.731   59893.4
 1.0  489.287       2.39401e5
 1.0  334.325  111773.0
 1.0  315.716   99676.4
 1.0  204.014   41621.9
 ⋮
 1.0  293.864   86356.2
 1.0  459.68        2.11305e5
 1.0  480.676       2.3105e5
 1.0  420.447  176776.0
 1.0  149.729   22418.8
 1.0  145.708   21230.7
 1.0  131.822   17377.0
 1.0  410.67        1.6865e5
 1.0  141.929   20143.9
 1.0  435.23        1.89425e5
 1.0  173.646   30152.9
 1.0  224.858   50561.1
```

In [51]:
```
1 betaHat = pinv(A)*yVals
```

Out[51]:
```
3-element Array{Float64,1}:
 891.2255677017974
  41.94486462276796
   0.2014071166238851
```

In [52]:
```
1 betaHat = A \ yVals
```

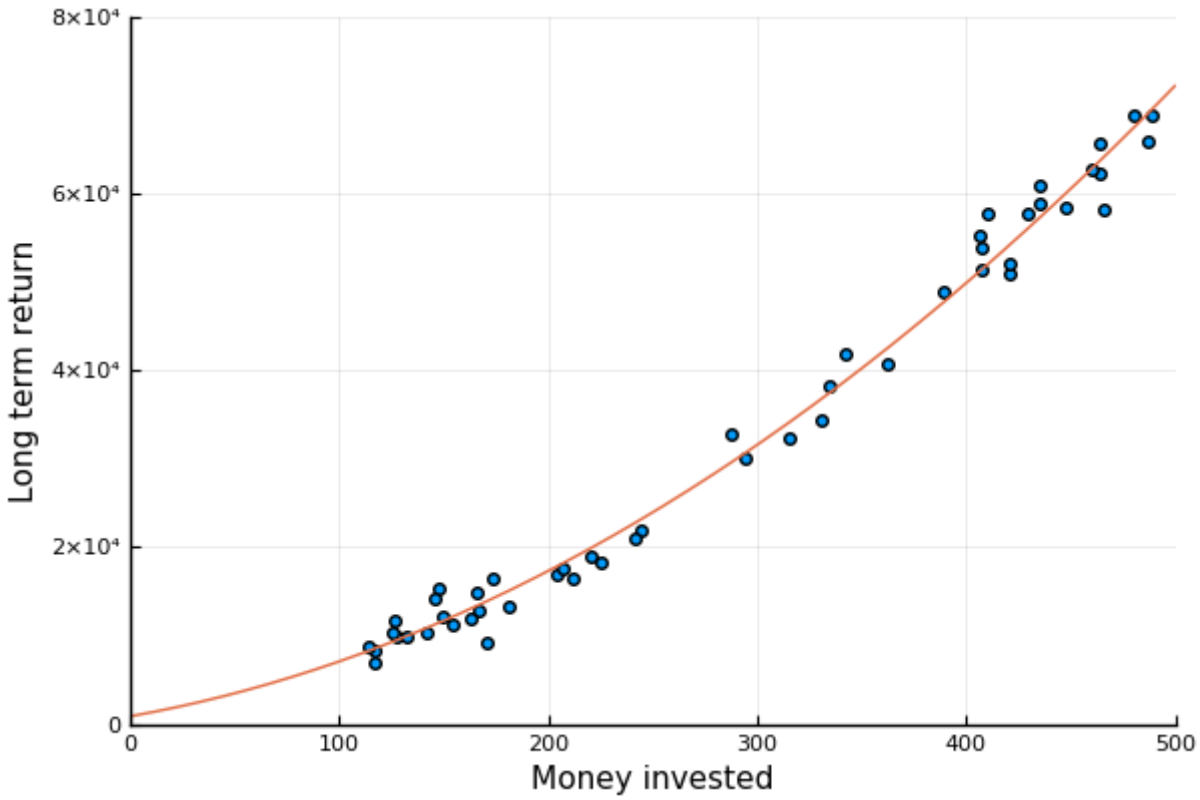
Out[52]:
```
3-element Array{Float64,1}:
 891.2255677022918
  41.94486462276237
   0.20140711662388516
```

In [53]:
```
1 yHat(x) = betaHat'*[1,x,x^2]
```

Out[53]:
```
yHat (generic function with 1 method)
```

In [54]:
```
1 xGrid = 0:500
2 yEst = yHat.(xGrid);
3 plot!(xGrid,yEst)
```

Out[54]:



What about a different model...

In [55]:
```
1 A =[xVals[i]^j for i in 1:n, j in 0:4];
2 betaHat = pinv(A)*yVals
```

Out[55]:
```
5-element Array{Float64,1}:
 569.9674662378493
 107.79862919485993
  -0.5453038951306075
   0.002571194871979982
  -2.732196407643429e-6
```

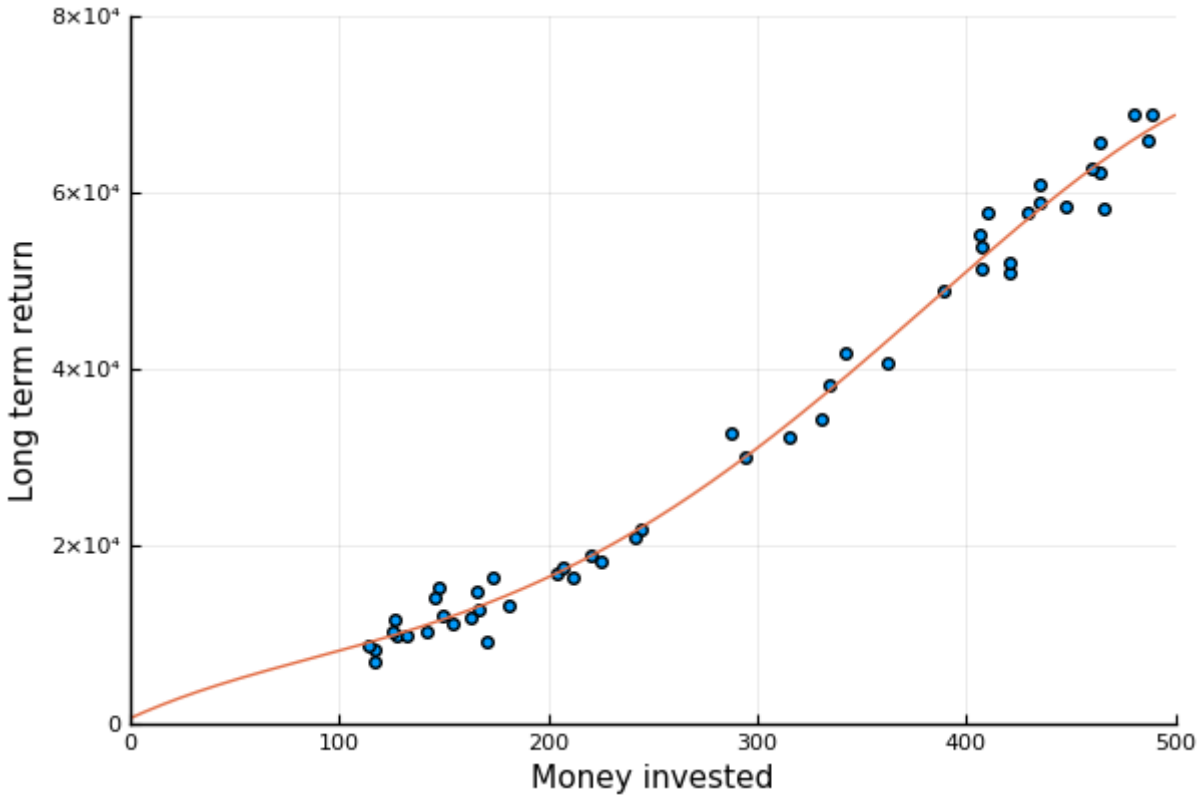
In [56]:
```
1 yHat(x) = betaHat'*[1,x,x^2,x^3,x^4]
```

Out[56]:
```
yHat (generic function with 1 method)
```

In [57]:

```
xGrid = 0:500
yEst = yHat.(xGrid);
scatter(xVals,yVals,legend=false,
    xlabel="Money invested", ylabel = "Long term return",xlim=(0,500),yl
plot!(xGrid,yEst)
```

Out[57]:



Getting a complex model...

In [58]:
```
powers = 0:0.1:3
A =[xVals[i]^p for i in 1:n, p in powers];
betaHat = pinv(A)*yVals
```

Out[58]:
```
31-element Array{Float64,1}:
       1.4794231724950373e7
       1.5035114691341102e7
       1.4631077774346262e7
       1.3287968815747976e7
       1.0960290111683741e7
       7.672479822751664e6
       3.610910037060839e6
 -841992.2900344953
      -5.106099906086788e6
      -8.455684959282205e6
      -1.014312818228975e7
      -9.597524065667778e6
      -6.672393640597768e6
       ⋮
      -6.006055603000715e6
      -8.379400639894038e6
      -4.486097589328948e6
       3.4657726184003353e6
       8.176590542220831e6
       3.1362095092576863e6
      -6.869136597689778e6
      -5.029653917860217e6
       9.071270217161536e6
      -4.440200719956398e6
  943134.7390007526
  -74512.00425352901
```

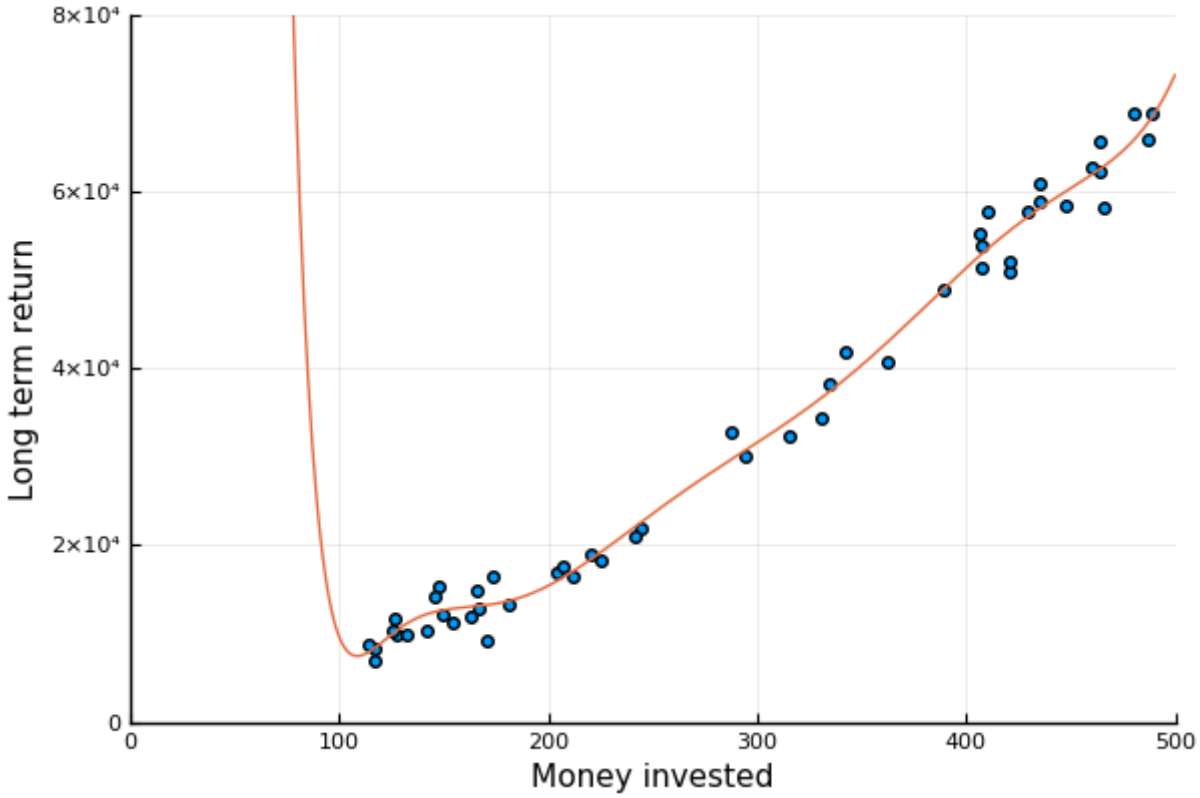
In [59]:
```
yHat(x) = betaHat'*[x^p for p in powers]
```

Out[59]:
```
yHat (generic function with 1 method)
```

In [60]:

```
xGrid = 0:500
yEst = yHat.(xGrid);
scatter(xVals,yVals,legend=false,
    xlabel="Money invested", ylabel = "Long term return",xlim=(0,500),yl
plot!(xGrid,yEst)
```

Out[60]:



Staying with the complex model, but regularizing

In [61]:
```
ridgeBeta(λ) = inv(A'*A + λ*I)*A'*yVals
λGrid = 0:20:2000
betaEsts = ridgeBeta.(λGrid);
norm.(betaEsts)
```

Out[61]:
```
101-element Array{Float64,1}:
   1.0694452386288696e10
 251.75550435588863
 169.01437722414056
 128.10957721247175
 101.919108748591
  85.79294572570862
  74.11582088144348
  65.33102525454333
  57.72364186699926
  52.26500705424
  47.79823652636996
  44.282063126446175
  40.97020791935847
   ⋮
  11.403819172830644
  11.36500714008561
  11.326548422847537
  11.286996279038512
  11.249971637359934
  11.216782244412785
  11.18416587755945
  11.154202748377083
  11.11454583424618
  11.08610907348081
  11.052163789391926
  11.022876528797354
```

In [62]:
```
1 betaHat = ridgeBeta(2000.0)
```

Out[62]:
```
31-element Array{Float64,1}:
 -0.034071245886825065
 -0.04619585373369618
 -0.06178550403882023
 -0.08134410109439166
 -0.10512031614919937
 -0.13284925208204593
 -0.1633392271726899
 -0.19389104141283137
 -0.21953751775545027
 -0.23211122582336996
 -0.2192658084076018
 -0.16367897484238503
 -0.0429160080441795
  ⋮
  3.4816676314254487
  3.5779442997922253
  2.9796468950077397
  1.5119682693645073
 -0.6836116196021473
 -2.899536366375301
 -3.7836988854436138
 -1.8763278558142238
  2.5059923033244544
  4.381060890888103
 -3.8411568758693004
  0.7799925785459643
```

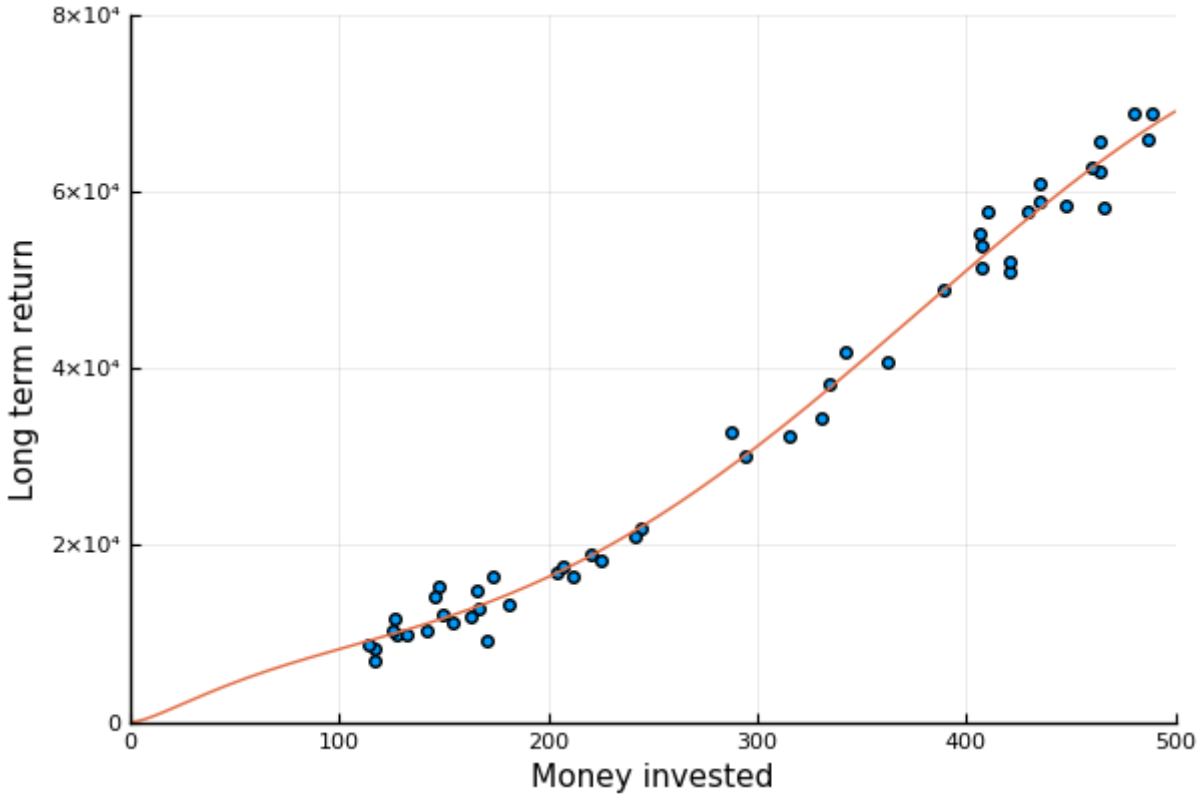
In [63]:
```
1 yHat(x) = betaHat'*[x^p for p in powers]
```

Out[63]:
```
yHat (generic function with 1 method)
```

In [64]:

```
xGrid = 0:500
yEst = yHat.(xGrid);
scatter(xVals,yVals,legend=false,
    xlabel="Money invested", ylabel = "Long term return",xlim=(0,500),yl
plot!(xGrid,yEst)
```

Out[64]:



In practice you may choose the "best" λ using cross-validation. We omit the details in this presentation.

A more advanced method is LASSO or Elastic Nets, where some variables are "knocked off" fully. We also omit the details.

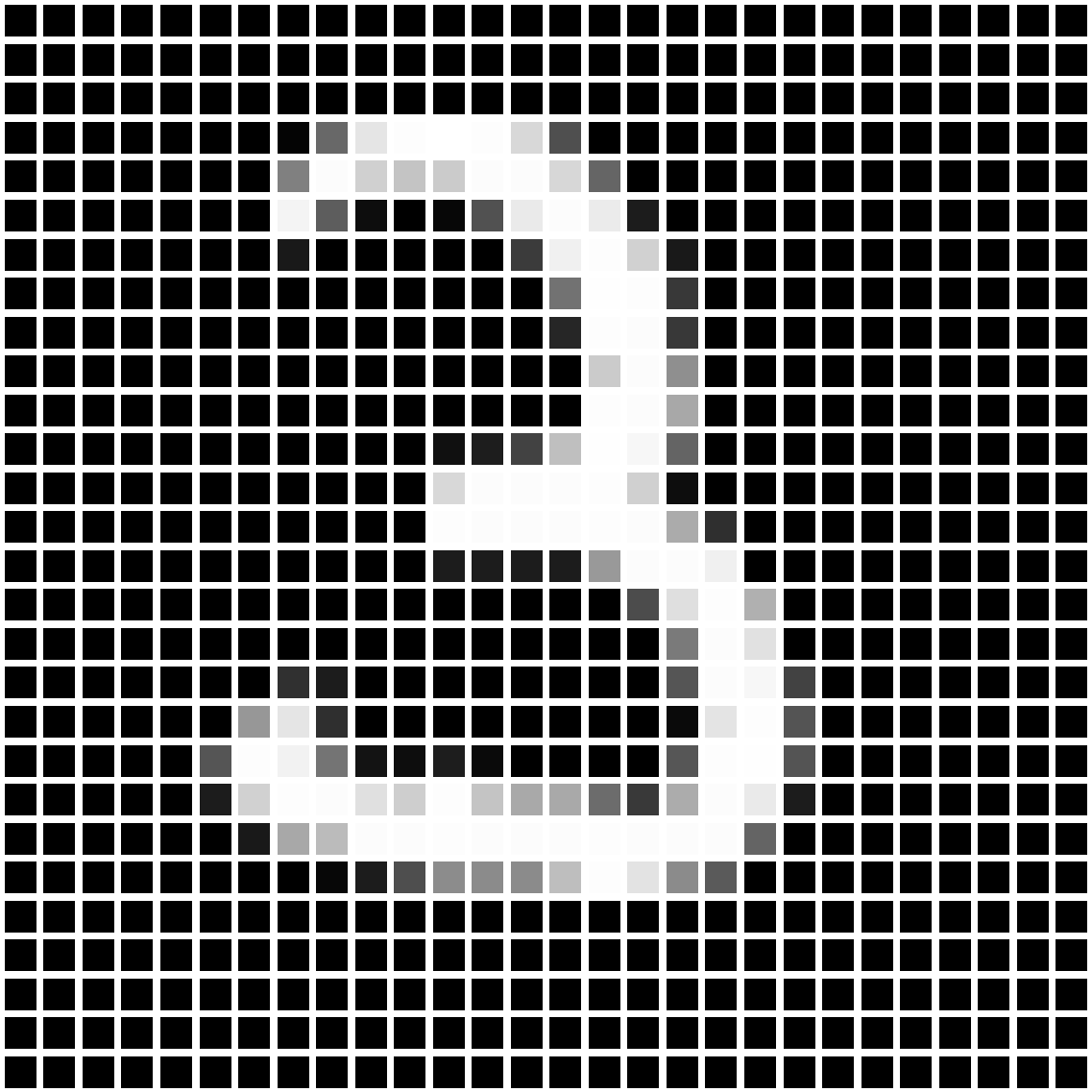
Classification

In [65]:
```
using Flux.Data.MNIST

images = MNIST.images(:train)
labels = MNIST.labels(:train)

train3 = images[labels .== 3]
train8 = images[labels .== 8]

length(train3),length(train8)
```

Out[65]: (6131, 5851)

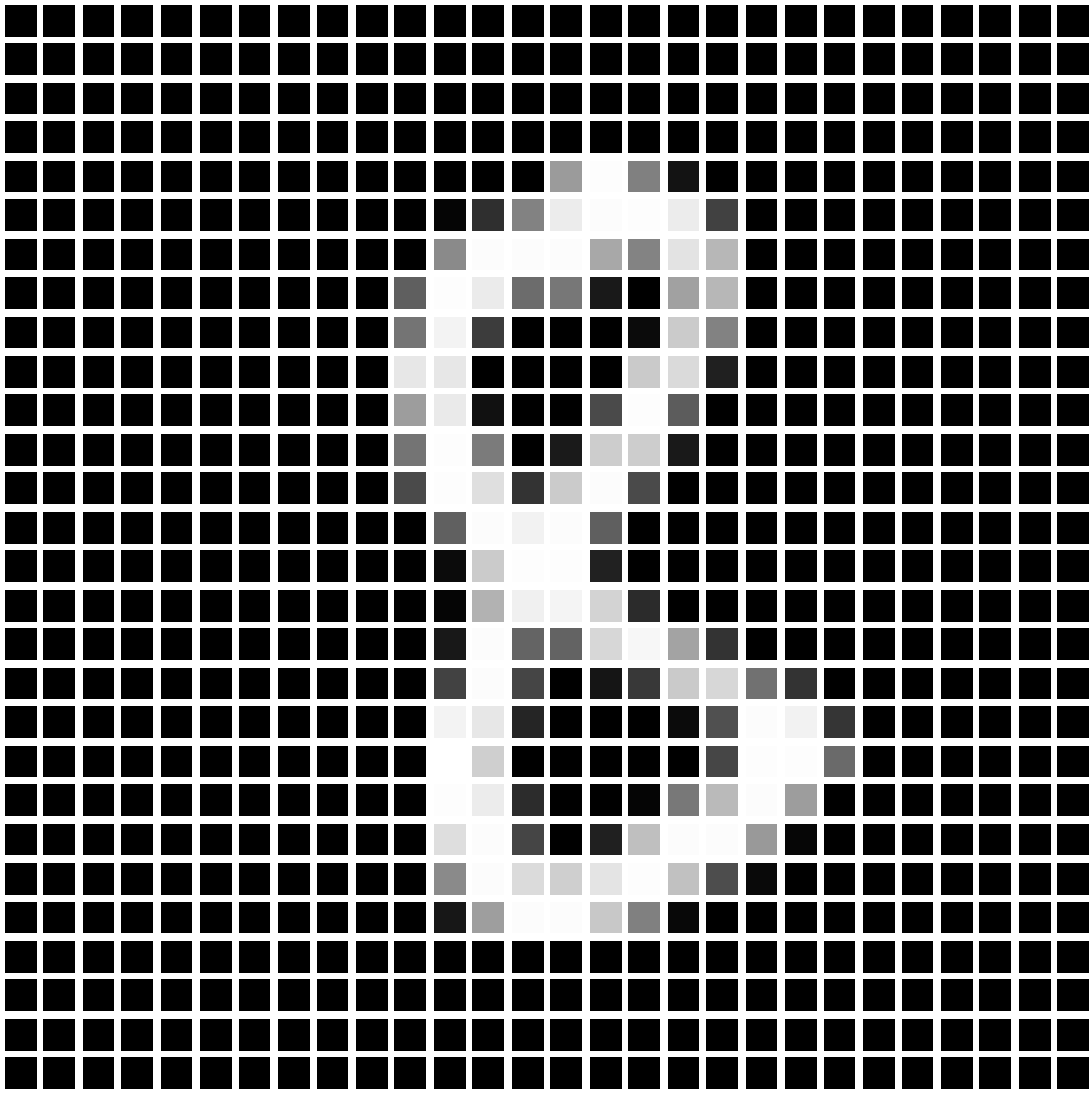
In [66]:
```
train3[54]
```

Out[66]:



In [67]:
```
1 train8[98]
```

Out[67]:



Also keep the test images

In [68]:
```
1 images = MNIST.images(:test)
2 labels = MNIST.labels(:test)
3
4 test3 = images[labels .== 3]
5 test8 = images[labels .== 8]
6
7 length(test3),length(test8)
```

Out[68]: (1010, 974)

In [69]:
```
1 length(train3),length(train8)
```

Out[69]: (6131, 5851)

In [70]:
```
1 yVals = [fill(-1,length(train3)) ; fill(+1,length(train8))]
```

Out[70]:
```
11982-element Array{Int64,1}:
 -1
 -1
 -1
 -1
 -1
 -1
 -1
 -1
 -1
 -1
 -1
 -1
 -1
  ⋮
  1
  1
  1
  1
  1
  1
  1
  1
  1
  1
  1
  1
  1
```

In [71]:
```
1 n = length(train3) + length(train8)
```

Out[71]:
```
11982
```

In [72]:

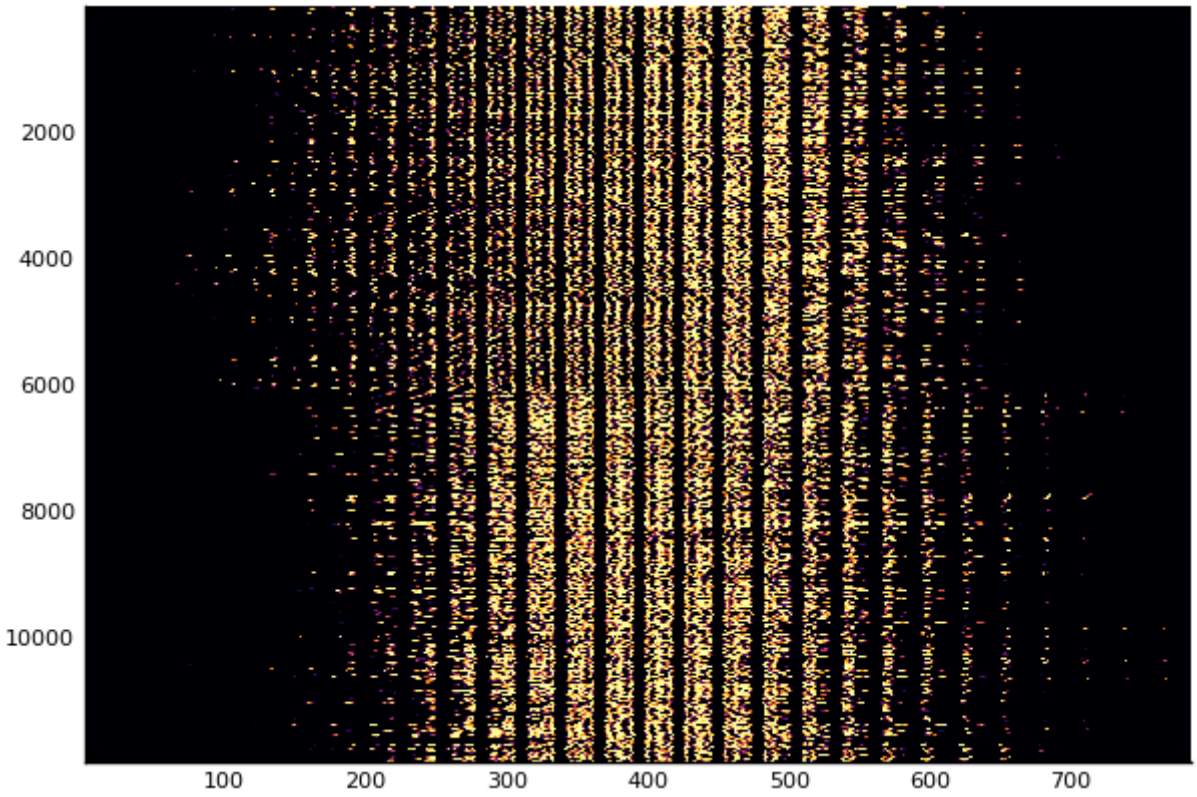
```
A = vcat(
    vcat([hcat(float.(t3)...) for t3 in train3]...),
    vcat([hcat(float.(t8)...) for t8 in train8]...)  )
A = [ones(n) A]
```

Out[72]:

```
11982×785 Array{Float64,2}:
 1.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  …  0.0  0.0  0.0  0.0  0.0  0.0
0.0
 1.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0     0.0  0.0  0.0  0.0  0.0  0.0
0.0
 1.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0     0.0  0.0  0.0  0.0  0.0  0.0
0.0
 1.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0     0.0  0.0  0.0  0.0  0.0  0.0
0.0
 1.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0     0.0  0.0  0.0  0.0  0.0  0.0
0.0
 1.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  …  0.0  0.0  0.0  0.0  0.0  0.0
0.0
 1.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0     0.0  0.0  0.0  0.0  0.0  0.0
0.0
 1.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0     0.0  0.0  0.0  0.0  0.0  0.0
0.0
 1.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0     0.0  0.0  0.0  0.0  0.0  0.0
0.0
 1.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0     0.0  0.0  0.0  0.0  0.0  0.0
0.0
 1.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  …  0.0  0.0  0.0  0.0  0.0  0.0
0.0
 1.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0     0.0  0.0  0.0  0.0  0.0  0.0
0.0
 1.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0     0.0  0.0  0.0  0.0  0.0  0.0
0.0
  ⋮                        ⋮              ⋱           ⋮
 1.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  …  0.0  0.0  0.0  0.0  0.0  0.0
0.0
 1.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0     0.0  0.0  0.0  0.0  0.0  0.0
0.0
 1.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0     0.0  0.0  0.0  0.0  0.0  0.0
0.0
 1.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0     0.0  0.0  0.0  0.0  0.0  0.0
0.0
 1.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  …  0.0  0.0  0.0  0.0  0.0  0.0
0.0
 1.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0     0.0  0.0  0.0  0.0  0.0  0.0
0.0
 1.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0     0.0  0.0  0.0  0.0  0.0  0.0
0.0
 1.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0     0.0  0.0  0.0  0.0  0.0  0.0
0.0
 1.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  …  0.0  0.0  0.0  0.0  0.0  0.0
0.0
 1.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0     0.0  0.0  0.0  0.0  0.0  0.0
0.0
```

In [73]:
```
1 heatmap(A,yflip = true, legend = false)
```

Out[73]:



In [74]:

```
@time begin
    betaHat = A \ yVals
end
```

```
  2.780118 seconds (4.97 k allocations: 79.289 MiB, 0.46% gc time)
```

Out[74]:

```
785-element Array{Float64,1}:
 -0.29874328285239965
  0.0
  0.0
  0.0
  0.0
  0.0
  0.0
  0.0
  0.0
  0.0
  0.0
  0.0
  0.0
  ⋮
  0.0
  0.0
  0.0
  0.0
  0.0
  0.0
  0.0
  0.0
  0.0
  0.0
  0.0
  0.0
```

In [75]:
```
sort(betaHat)
```

Out[75]:
```
785-element Array{Float64,1}:
 -447.77470256684035
  -91.83906252131727
  -20.93420668900583
  -20.12281731295012
  -12.970096219801018
   -8.585745051595348
   -7.5718682955670324
   -4.522049842472704
   -3.7097037552105765
   -3.182659459842445
   -3.166098424169005
   -1.8570910582374687
   -1.8022722589018405
    ⋮
    4.658930086917823
    5.1630948218363635
    5.853308051783189
    6.170218028440353
    6.922854776188601
    6.956089738575784
    8.054060105724242
   12.39920003810355
   20.872658410985984
   30.893504367594304
   37.68300377010177
  415.66111581860565
```

In [76]:
```
decide(img) = betaHat'*[1 ; vcat(float.(img)...)] > 0 ? 8 : 3
```

Out[76]:
```
decide (generic function with 1 method)
```

Let's try it out...

In [77]:
```
decide(test8[1]),
decide(test3[1]), #makes a mistake...
decide(test3[2])
```

Out[77]:
```
(8, 8, 3)
```

In [78]:
```
truePositive8 = sum(decide.(test8) .== 8)
truePositive3 = sum(decide.(test3) .== 3)

falsePositive3 = sum(decide.(test8) .== 3)
falsePositive8 = sum(decide.(test3) .== 8);
```

In [79]:
```
decisionMat = [truePositive3  falsePositive8; falsePositive3 truePositiv
```

Out[79]:
```
2×2 Array{Int64,2}:
 968   42
  40  934
```

In [80]:
```
1 nn = [length(test3),length(test8)]
```

Out[80]:
```
2-element Array{Int64,1}:
 1010
  974
```

In [81]:
```
1 decisionMat ./ nn
```

Out[81]:
```
2×2 Array{Float64,2}:
 0.958416   0.0415842
 0.0410678  0.958932
```

We see that we achive about 95% accuracy! The next step is to consider multi-class classification (deciding between the digits '0'--'9')... see for example the "Statistics with Julia" book by Hayden Klok and Yoni Nazarathy.

Thank You