This is the front page of the practice exam. When you come to the real exam, you'll see here a place to write your name and student number. The duration of the final exam is 180 minutes. The final exam is of similar question composition and format to this practice exam.

Instructions:

There are 5 questions, with 20 points each. Each question has two items. Item (a) worth 14 points and item (b) worth 6 points.

Answer all questions on this exam paper.

Use the other side of the paper if additional space is required.

Question 1

(a) Consider the data matrix,

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix},$$

with Singular Value Decomposition (SVD) of the form

$$A = U \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix} V^T,$$

where U is a 4×2 matrix with orthonormal columns, and V is a 2×2 matrix with orthonormal columns. Determine the singular values σ_1 and σ_2.

(b) Consider now the matrix $B = A\,A^T\,A$. What are the singular values of B?

Question 2

(a) Assume you are presented with tuples of data points $(x_{11}, x_{21}, G_1, y_1), \ldots, (x_{1n}, x_{2n}, G_n, y_n)$. Where x's and y's are real values and $G_i \in \{\text{`m'}, \text{`f'}\}$ where 'm' stands for 'male' and 'f' stands for 'female'. To describe this data, you wish to fit a model,

$$y = \sqrt[3]{\begin{cases} \beta_m + \beta_1 x_1 + \beta_2 x_2 & \text{if male,} \\ \beta_f + \beta_1 x_1 + \beta_2 x_2 & \text{if female.} \end{cases}}$$

You do this by selecting $\beta = [\beta_0, \beta_1, \beta_2, \beta_3]^T$ that will minimize,

$$L = \sum_{i=1}^{n} \left({y_i}^3 - \beta_0 - \beta_1 x_{1i} - \beta_2 x_{2i} - \beta_3 I_i \right)^2,$$

where $I_i = 0$ if $G_i =$ ''m' and $I_i = 1$ if $G_i =$ 'f'. You then set $\beta_m = \beta_0$ and $\beta_f = \beta_0 + \beta_3$.
To minimize L, you set $\hat{\beta} = (A^T A)^{-1} A^T v$ where $A \in \mathbb{R}^{n \times 4}$ and $v \in \mathbb{R}^n$. Assume that n is even and the data is sorted such that the first $1, \ldots, n/2$ observations are 'male' and the others 'female'. Determine A and v.

(b) Say that when you attempt to compute $(A^T A)^{-1}$ you find out that A does not have linearly independent columns and thus $A^T A$ is singular. You then resort to ridge regression by aiming to minimize,

$$||A\beta - b||^2 + \lambda||\beta||^2.$$

with $\lambda > 0$. The minimizer is given by $\hat{\beta}_\lambda = (A^T A + \lambda I)^{-1} A^T v$. Does the inverse in this formula exist for any $\lambda > 0$ or are there cases where it doesn't? Prove your claim.

Question 3

(a) Consider the vectors $v_1 = [1, 0, 1]^T$, $v_2 = [6, 6, 0]^T$ and the 3×2 matrix $A = [v_1\ v_2]$. Determine a QR factorization of A having the form $A = QR$ where Q is a 3×2 matrix with orthonormal columns and R is an upper triangular matrix. Throughout this question, avoid using decimal points, but rather use exact arithmetic.

(b) Assume you now don't know the values in A and only have the values in R that you computed above. Use the values of R to determine the eigenvalues of the matrix AA^T.

Question 4

(a) Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$, given by $f(x) = (x-d)^T C\,(x-d) + b^T x$ with,

$$C = \begin{bmatrix} 4 & 2 \\ 2 & -2 \end{bmatrix}, \qquad b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad \text{and} \qquad d = \begin{bmatrix} 6 \\ 4 \end{bmatrix}.$$

Compute the gradient function $\nabla f(x)$ and determine a point $x \in \mathbb{R}^2$ for which x, $\nabla f(x) = 0$. Is this a local minimum point?

(b) Consider now a least squares problem, minimizing $||Ax - b||$ where $A \in \mathbb{R}^{n \times 2}$ and $b \in \mathbb{R}^n$. Observe the following computation (completion of the square),

$$\begin{aligned}
||Ax - b||^2 &= (Ax - b)^T(Ax - b) \\
&= x^T A^T Ax - 2b^T Ax + b^T b \\
&= x^T A^T Ax - 2b^T Ax + b^T b - 2b^T A^T Ax + 2b^T A^T Ax \\
&= x^T A^T Ax - 2b^T A^T Ax + b^T b + 2b^T A^T Ax - 2b^T Ax \\
&= (x - b)^T A^T A(x - b) + 2b^T(A^T A - A)x.
\end{aligned}$$

Is there a matrix A and vector b such that the least square problem is equivalent to minimizing $f(\cdot)$ from (a)? If so, express A and b in terms of C, D and d, otherwise, explain why there isn't such an A.

Question 5

(a) Consider the matrices,

$$A = \begin{bmatrix} 1/2 & 1 \\ 0 & 2 \end{bmatrix}, \qquad \text{and} \qquad B = \begin{bmatrix} 2 & -1 \\ 0 & 1/2 \end{bmatrix}.$$

Determine the eigenvalues of the matrix $B^{20}A^{22}$.

(b) Consider now the 4×4 matrix W,

$$W = \begin{bmatrix} B^{20}A^{22} & B^{20}A^{7} \\ 0 & B^{20}A^{19} \end{bmatrix}$$

Determine the eigenvalues of W.

END OF EXAMINATION