Markovian (and Deterministic) Dynamical Systems

Linear dynamical systems

Let x_1, x_2, . . . be a sequence of n-vectors. In linear dynamical systems we consider x_t as the current state. x_{t+1} as the next state, x_{t-1} as the previous state. Then we can represent x_{t+1} of x_t as:

$$x_{t+1} = A_t x_t,\ t = 1, 2\cdots$$

In this function, A_t is the dynamics matrices. We can use this function to calculate the future state with the current state.

Markov chain

Markov chain studied the probability that the next state value only depends on the current state but not on the previous state values, we can describe the function as:

$$P_{\Gamma}(X_{n+1} = x|\ X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n) = P_r(X_{n+1} = x|X_n = x_n)$$

The Markov chain is independent of the initial distribution $P_{\Gamma}(X_1 = x_1)$. In Markov chain, the transition probabilities P_{ij} is the probability of state j at time n moving to the state i at time n+1: $P_{\text{ij}} = P_{\Gamma}(X_{n+1} = x_i|X_n = x_j)$. The transition matrix P contains all these probabilities. This matrix applies to the transition from state x(n) to state x(n+1) every time, we have:

$$X_{n+1} = \begin{bmatrix} P_{\Gamma}(X_{n+1} = 1) \\ \vdots \\ P_{\Gamma}(X_{n+1} = N) \end{bmatrix} = \begin{bmatrix} P_{11} & \cdots & P_{1N} \\ \vdots & \ddots & \vdots \\ P_{N1} & \cdots & P_{NN} \end{bmatrix} \begin{bmatrix} P_{\Gamma}(X_n = 1) \\ \vdots \\ P_{\Gamma}(X_n = N) \end{bmatrix}$$

In this equation, every P_{ij} should satisfy that 0<P_{ij}<1 and each column adds to 1.

Steady state

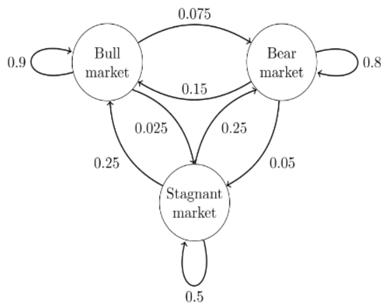
Continuing to multiply the initial distribution(y0) by the Markov matrix (yn = Pny0, as n→ ∞), the product converges to the steady state. The steady state of a Markov chain, a column vector u, must satisfy the following equation, Pu = u. This equation can be rewritten as Pu = 1u. So the steady state of a Markov matrix is just the eigenvector of the matrix corresponding to the eigenvalue λ= 1.

Q-learning and MDP

There are many techniques used to gain information and make decisions simultaneously in Reinforcement Learning. Markov Decision Processes (MDP) and Q-learning are one of those techniques based on the Markov chain.

Application

Markov chains have many applications in statistics. For example, the Markov chain can predict the trend in the stock market. The states represent whether a hypothetical stock market have a bull market, bear market, or stagnant market trend during a given week. The picture on the left shows the probability of the transition between each state, then we can put those probabilities into the transition matrix



$$P = \begin{bmatrix} 0.9 & 0.075 & 0.025 \\ 0.15 & 0.8 & 0.05 \\ 0.25 & 0.25 & 0.5 \end{bmatrix}$$

Using the transition matrix above, we can predict the trend of the stock market, if the given initial state is the stagnant market, we can calculate the percentage of different future states in n weeks, after the calculation, the matrix would become steady after n times, that 62.5% of weeks will be in a bull market, 31.25% of weeks will be in a bear market and 6.25% of weeks will be stagnant, since:

$$\lim_{N\to\infty} (P)^N = \begin{bmatrix} 0.625 & 0.3125 & 0.0625 \\ 0.625 & 0.3125 & 0.0625 \\ 0.625 & 0.3125 & 0.0625 \end{bmatrix}$$