MATH7502 Group project: Signal Processing

1. Convolution

Convolution is a mathematical operation on two functions that produces a third function expressing how the shape of one is modified by the other.

Vector convolution: The convolution of an n-vector a and an m-vector b is the (n+m-1)-vector denoted c = a*b, with the entries,

$$c_k = \sum_{i+j=k+1} a_i b_j, \quad k = 1,\dots,n+m-1$$

Matrix convolution: The convolution of an m×n matrix A and a p×q matrix B is the (m+p-1)×(n+q-1) matrix C, with the entries,

$$C_{rs} = \sum_{i+k=r+1,\, j+l=s+1} A_{ij} B_{kl}, \quad r = 1,\dots,m+p-1,\ s = 1,\dots,n+q-1$$

2. Fourier transform

Discrete Fourier Transform (DFT): Given a complex N-vector $f = (f_0, f_1, \dots, f_{N-1})$, DFT takes f in "x-space" to its coefficients in "frequency space",

$$f = c_0 b_0 + c_1 b_1 + \dots + c_{N-1} b_{N-1},$$

where b_k are N basis vectors with $(b_k)_j = e^{2\pi ijk/N} = (e^{2\pi i/N})^{jk}$ and c_k are discrete Fourier coefficients.

Two N×N matrices, Fourier Matrix F and DFT Matrix Ω, are defined for actual calculation. F contains powers of $w = e^{2\pi i/N}$ and Ω contains powers of $\omega = e^{-2\pi i/N} = \overline{w}$.

DFT: Ω times f produces the discrete Fourier coefficients c.

Inverse - DFT: F times c bring back the vector f.

Fast Fourier Transform (FFT): To multiply F times c as quickly as possible, FFT is to connect F_N with the half-size Fourier matrix $F_{N/2}$,

$$F_N = \begin{bmatrix} I_{N/2} & D_{N/2} \\ I_{N/2} & -D_{N/2} \end{bmatrix} \begin{bmatrix} F_{N/2} & \\ & F_{N/2} \end{bmatrix} \begin{bmatrix} even - odd \\ permutatio\ n \end{bmatrix}$$

Where I is the identity matrix, D is the diagonal matix. Keep on this step going to $F_{N/4}, F_{N/8}, \dots$,the final count for size $N = 2^l$ is reduced from N^2 to $\frac{1}{2}Nl$ (Full FFT).

3. Shift matrices and Circulant Matrices

Upward shift cyclic permutation: When n×n matrix P multiplies a n n-vector x, the components of x shift upward. P is a binary matrix with ones only on the superdiagonal and $P_{n,1}$, and zeroes elsewhere.

Circulant matrix: Given an n vector $c = (c_0, c_1, \dots, c_{n-1})$, its circulant matrix C is defined as,

$$C = c_0 P^n + c_1 P^1 + c_2 P^2 + \dots + c_{n-1} P^{n-1}$$

Cyclic convolution: Given two n-vectors c and d, respectively generate their circulant matrices C and D. First row of product of C and D is the cyclic convolution of c and d, denoted as $c \otimes d$.

Convolution Rule: $F(c \otimes d) = (Fc).*(Fd)$

LHS: Cyclic convolve c and d first, then transform by F.

RHS: Transform c and d by F first, then multiply Fc times Fd component by component.