Fourier Transforms

Fourier Transforms are a way to represent waveforms as a linear combination of periodic basis functions. These basis functions are orthogonal, allowing for simplicity and guaranteeing linear independence. Discrete Fourier Transforms in particular deal with discrete N-vectors, mapping them from x space (usually time) to frequency space, using two special $N \times N$ **Vandermonde** matrices: The Fourier Matrix (F) (composed of N-basis vectors b_k where $(b_k)_j = e^{(2\pi i/N)jk}$), and its **functional** inverse: The DFT Matrix (Ω). This allows representing complex signals as linear combinations of sinusoidal functions, using Euler's formula: $e^{-i2\pi/N} = \cos(2\pi/N) - i \cdot \sin(2\pi/N)$. Practical applications include audio filters (noise reduction, high/low pass) and image transformations such as edge detection and blur.

$$\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega^1 & \omega^2 & \cdots & \omega^{(k-1)} \\ 1 & \omega^2 & \omega^4 & \cdots & \omega^{2\cdot(k-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{(j-1)} & \omega^{(j-1)\cdot 2} & \cdots & \omega^{(j-1)(k-1)} \end{bmatrix} = \begin{cases} \Omega_{\mathrm{N\times N}} \Leftrightarrow \omega = e^{-2\pi ijk/N} \\ F_{N\times N} \Leftrightarrow \omega = e^{2\pi ijk/N} \end{cases}$$

Figure 1 Fourier Transform Matrices

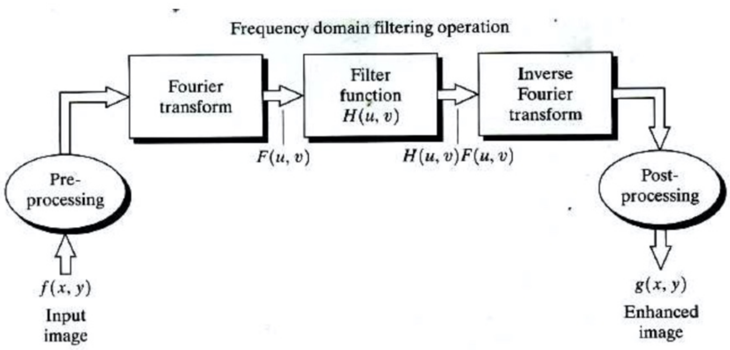

Figure 2 Fourier Transform Typical Process Flow

Convolution

In signal processing convolution can be described as the result of an input signal being modified by the impulse response of a system, for example the effect of transmitting sound through water.

The convolution of f and g $(f * g)$ can be expressed the area under the curve of both functions as the reversed function g is shifted over function f.

$$(f * g)(t) = \int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau$$

An image can be convolved using matrix convolution where A is the original image, and B represents the image filter or kernel.

Convolution can also be applied to vectors and matrices. A is a $m \times n$ matrix, B is a $p \times q$ matrix and C is the $(m+p-1) \times (n+q-1)$ convolution matrix.

$$C = A * B,\ \ C_{rs} = \sum_{i+k=r+1, j+l=s+1} A_{ij}B_{kl}$$

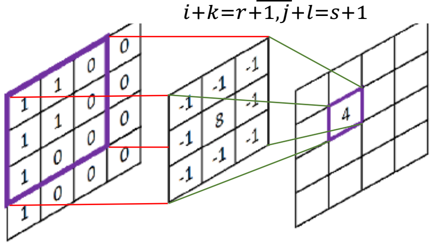

Figure 3 Convolution Filter: Edge Detection

Example applications

- Smoothing time series data (convolution can also be interpreted as a weighted average)
- Image processing such as edge detection, sharpening and blurring
- Convolutional neural networks are used for image classification and analysis