Constrained Optimization

Group members: Xinyi Hu; Yiting Liu; Zhiwei Hu

Definition: In mathematical optimization, constrained optimization is the process of optimizing an objective function with respect to some variables in the presence of constraints on those variables.

Constrained Least Squares:

Definition: The constrained least squares problem combines the problems of solving a set of linear equations with the least squares problems.

Format: A general constrained minimization problem may be written as follows:

$$\text{Minimize } \|Ax - b\|^2$$
$$\text{Subject to } Cx = d$$

Solution Methods:

(1)Lagrange Multipliers:

For the case of only one constraint and only two choice variables consider the optimization problem

$$\text{maximize } f(x, y) \text{ subject to } g(x, y) = 0.$$

We assume that both f and g have continuous first partial derivatives. We introduce a new variable (λ) called a Lagrange multiplier (or Lagrange undetermined multiplier) and study the Lagrange function (or Lagrangian or Lagrangian expression) defined by

$$\mathcal{L}(x, y, \lambda) = f(x, y) - \lambda g(x, y),$$

Then we set the derivatives $\frac{\partial \mathcal{L}}{\partial x}$ and $\frac{\partial \mathcal{L}}{\partial y}$ and $\frac{\partial \mathcal{L}}{\partial \lambda}$ $to\ zero$.Solve these equations for x, y, λ will gain the maximize of f (x, y). And this method of can be extended to solve problems with multiple constraints using a similar argument.

$$\mathcal{L}(x_1, \dots x_n, \lambda_1, \dots, \lambda_M) = f(x_1, \dots x_n) - \sum_{k=1}^{M} \lambda_k g_k(x_1, \dots x_n)$$

$$\nabla_{x_1,\dots,x_n,\lambda_1,\dots,\lambda_M} \mathcal{L}(x_1, \dots, x_n, \lambda_1, \dots, \lambda_M) = 0 \iff \begin{cases} \nabla f(\mathbf{x}) - \sum_{k=1}^{M} \lambda_k \nabla g_k(\mathbf{x}) = 0 \\ g_1(\mathbf{x}) = \cdots = g_M(\mathbf{x}) = 0 \end{cases}$$

(2)QR Factorization:

Given the KKT equations, it can be rewrite as

$$2(A^T A + C^T C)\hat{x} + C^T w = aA^T b,$$

where $w = \hat{z} - 2d$. By solving the function

$$R\hat{x} = Q_1^T b - (1/2) Q_2^T w$$

We can get the solution of $\hat{x}$.

Application:

A time-varying linear dynamical system:$X_{t+1} = A_t x_t + B_t u_t, t = 1,2, \dots \dots$. This system has an output:$y_t = C_t\, x_t\, , t = 1,2 \dots \dots$ Linear quadratic control refer to the problem of choosing the input and state sequence, so as to minimize a sum of squares objective, subject to the dynamics equation, the output equations and additional linear equality constraints.

The objective function $J = J_{\text{output}} + \rho J_{\text{input}},$

$$\begin{aligned} J_{\text{output}} &= \|y_1\|^2 + \cdots + \|y_T\|^2 = \|C_1 x_1\|^2 + \cdots + \|C_T x_T\|^2, \\ J_{\text{input}} &= \|u_1\|^2 + \cdots + \|u_{T-1}\|^2. \end{aligned}$$

$$\begin{array}{ll} \text{minimize} & J_{\text{output}} + \rho J_{\text{input}} \\ \text{subject to} & x_{t+1} = A_t x_t + B_t u_t, \quad t = 1, \dots, T-1, \\ & x_1 = x^{\text{init}}, \quad x_T = x^{\text{des}}, \end{array}$$

We can solve this question by changing it into least squares formulation.