Question 1

(a) You may use the singular value decomposition (SVD) to answer this question. Consider a 1000×50 data matrix A, summarizing data of 1000 individuals and 50 features with $\text{rank}(A) = 49$. Denote the eigenvalues of AA^T by $\lambda_1, \lambda_2, \ldots$ with $\lambda_i \geq \lambda_j$ for $j > i$. What is λ_{50}? Explain your answer.

Because $\text{rank}(A) = 49$ the svd has singular values $\sigma_1, \sigma_2, \ldots, \sigma_{49} > 0$ where $\sigma_i = \sqrt{\lambda_i}$ and the last eigenvalue of AA^T is 0.

(b) Assume now that $\lambda_1 = \lambda_2 = \ldots = \lambda_{49} = 4$. Determine,

$$\sqrt{\sum_{i=1}^{1000}\sum_{j=1}^{50} A_{ij}^2}.$$

This is the Frobenius norm and it holds that $\|A\|_F^2 = \sum\sum A_{ij}^2$ satisfies

$$\|A\|_F^2 = \sigma_1^2 + \cdots + \sigma_{49}^2 = 4 \cdot 49 = 196$$

Hence $\|A\|_F = \sqrt{196} = 14$

Question 2

(a) Assume you are presented with pairs of data points $(x_1, y_1), \dots, (x_n, y_n)$ where $x_i \neq x_j$ for $i \neq j$. To describe this data, you wish to fit a model,

$$y = \beta_0 + \beta_1 x + \beta_2 x^2,$$

by selecting $\beta = [\beta_0, \beta_1, \beta_2]^T$ that will minimize,

$$\sqrt{\sum_{i=1}^{n} \left(y_i - \beta_0 - \beta_1 x_i - \beta_2 x_i^2\right)^2}.$$

You do this by setting $\hat{\beta} = (A^T A)^{-1} A^T v$ where $A \in \mathbb{R}^{n \times 3}$ and $v \in \mathbb{R}^n$. Determine A and v.

Set

$$A = \begin{bmatrix} 1 & X_1 & X_1^2 \\ 1 & X_2 & X_2^2 \\ \vdots & \vdots & \vdots \\ 1 & X_n & X_n^2 \end{bmatrix} \quad \text{and} \quad v = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

then $\|A\beta - v\| = \sqrt{\sum_{i=1}^{n} \left(y_i - \beta_0 - \beta_1 X_i - \beta_2 X_i^2\right)^2}$

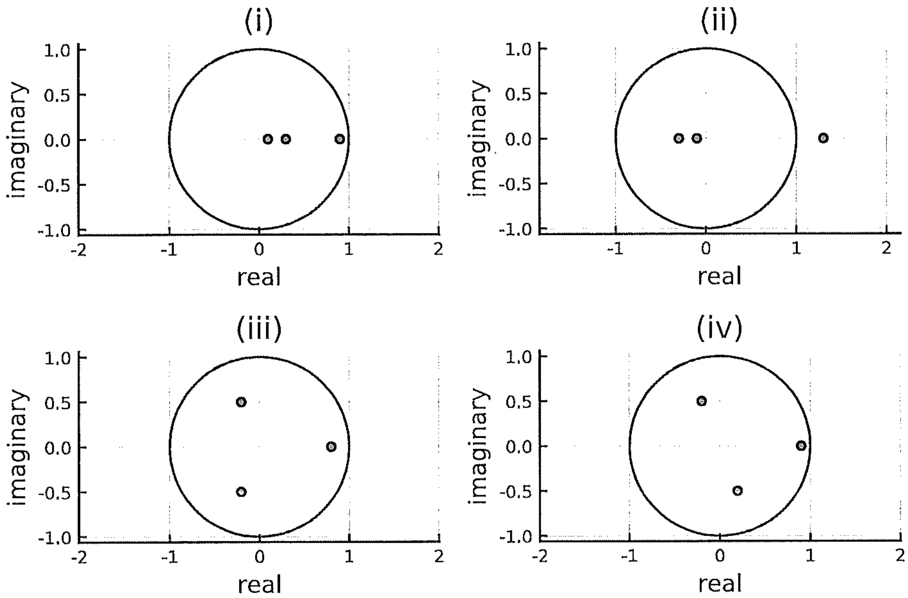
This least squares problem is solved by $\hat{\beta} = (A^T A)^{-1} A^T v$ and A is with independent collumns because $X_i \neq X_j$ and thus $(A^T A)^{-1}$ exists

(b) Assume now that you wish to find $\hat{\beta}$ using gradient descent. For this you compute the gradient descent step with learning rate $\eta > 0$, via,

$$\beta_{t+1} = \beta_t - \eta 2A^T(A\beta_t - v) = (I - 2\eta A^T A)\beta_t + 2\eta A^T v = G(\eta)\beta_t + 2\eta A^T v,$$

where the matrix $G(\eta) = I - 2\eta A^T A$ depends on the learning rate. You set the learning rate at a low enough value, η^* such that $\beta_t \to \hat{\beta}$ (gradient descent converges).

Let the eigenvalues of $G(\eta^*)$ be λ_1, λ_2 and λ_3. Below are four alternative plots of λ_1, λ_2 and λ_3 on the complex plane. For each them, determine if it is possible or not and if not, explain why.



(i) Possible / Impossible: G() is symmetric hence has real eigenvalues, ~~with~~ Further spectral radius < 1 due to convergence

(ii) Possible / Impossible: Because in this case $\beta_t \not\to \hat{\beta}$

(iii) Possible / Impossible: complex eigenvalues cannot be for real matrix.

(iv) Possible / Impossible: Not complex-conjugate. Cannot be for real characteristic polynomial

Question 3

(a) Consider the vectors $v_1 = [1,1,0,0]^T$, $v_2 = [1,1,1,1]^T$ and the 4×2 matrix $A = [v_1\ v_2]$. Determine a QR factorization of A having the form $A = QR$ where Q is a 4×2 matrix with orthonormal columns and R is an upper triangular matrix. Choose Q and R such that neither have negative entries. Throughout this question, avoid using decimal points, but rather use exact arithmetic.

$$\tilde{q}_1 = v_1 \quad q_1 = \frac{\tilde{q}_1}{\|\tilde{q}_1\|} = \left[\tfrac{1}{\sqrt{2}}\ \tfrac{1}{\sqrt{2}}\ 0\ 0\right]^T$$

$$\tilde{q}_2 = v_2 - (v_2^T q_1) q_1 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix} - \frac{2}{\sqrt{2}}\begin{bmatrix}\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\\0\\0\end{bmatrix} = \begin{bmatrix}0\\0\\1\\1\end{bmatrix}$$

$$q_2 = \frac{\tilde{q}_2}{\|\tilde{q}_2\|} = \left[0\ \ 0\ \ \tfrac{1}{\sqrt{2}}\ \ \tfrac{1}{\sqrt{2}}\right]$$

$$Q = \begin{bmatrix} q_1 & q_2 \end{bmatrix}$$

$$R_{11} = \|\tilde{q}_1\| = \sqrt{2} \qquad R_{12} = q_1^T v_2 = \frac{2}{\sqrt{2}} = \sqrt{2}$$

$$R_{21} = 0 \qquad R_{22} = \|\tilde{q}_2\| = \sqrt{2}$$

Hence

$$\underbrace{\begin{bmatrix}1 & 1\\1 & 1\\0 & 1\\0 & 1\end{bmatrix}}_{A} = \underbrace{\begin{bmatrix}\frac{1}{\sqrt{2}} & 0\\\frac{1}{\sqrt{2}} & 0\\0 & \frac{1}{\sqrt{2}}\\0 & \frac{1}{\sqrt{2}}\end{bmatrix}}_{Q} \underbrace{\begin{bmatrix}\sqrt{2} & \sqrt{2}\\0 & \sqrt{2}\end{bmatrix}}_{R}$$

(b) Let $b = [0\ 1\ 1\ 0]^T$. You now wish to find an approximate solution to $Ax = b$ in the sense that x minimizes $||Ax - b||$. Use your QR factorization from the previous exercise to find x.

We have $\hat{x} = R^{-1} Q^T b$

$$\hat{x} = \underbrace{\frac{1}{\sqrt{2}\sqrt{2} - 0\sqrt{2}} \begin{bmatrix} \sqrt{2} & -\sqrt{2} \\ 0 & \sqrt{2} \end{bmatrix}}_{R^{-1}} \underbrace{\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}}_{Q^T} \underbrace{\begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}}_{b}$$

$$= \frac{1}{2} \begin{bmatrix} \sqrt{2} & -\sqrt{2} \\ 0 & \sqrt{2} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 1 - 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix}$$

Question 4

(a) Consider a collection of N, n-vectors on which you execute the k-means algorithm. Denote the vectors by $x_1, \dots, x_N$ and denote by G_j the set of indexes of vectors in group j resulting from k-means. Thus for example if $G_3 = \{17, 21, 302\}$ then the vectors x_{17}, x_{21} and x_{302} make up group 3. Denote,

$$z_j = \frac{1}{|G_j|} \sum_{i \in G_j} x_j, \qquad \text{for} \qquad j = 1, \dots, k.$$

In general, k-means is only a heuristic and it's solution does not always exactly minimize the clustering objective,

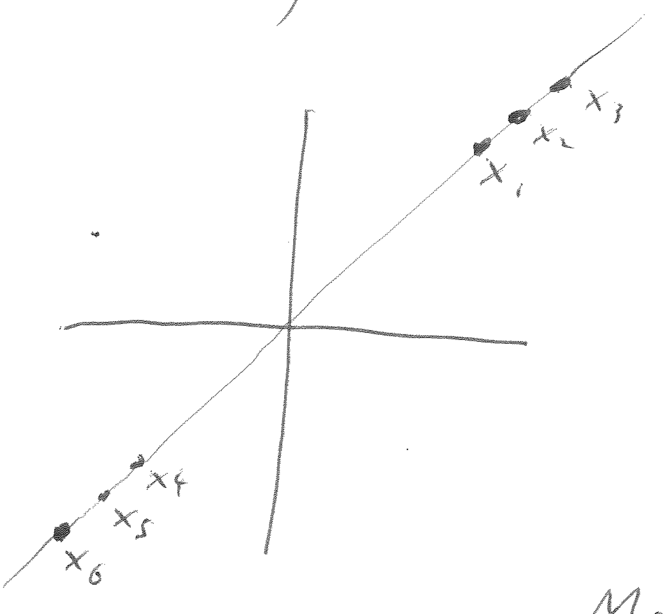
$$J = \sum_{j=1}^{k} \sum_{i \in C_j} ||x_i - z_j||^2.$$

However, sometimes it does. For part (a), assume that $n = 2$, $k = 2$ and $N = 6$ with,

$$x_1 = \begin{bmatrix} 9 \\ 9 \end{bmatrix}, \quad x_2 = \begin{bmatrix} 10 \\ 10 \end{bmatrix}, \quad x_3 = \begin{bmatrix} 11 \\ 11 \end{bmatrix}, \quad x_4 = \begin{bmatrix} -9 \\ -9 \end{bmatrix}, \quad x_5 = \begin{bmatrix} -10 \\ -10 \end{bmatrix}, \quad x_6 = \begin{bmatrix} -11 \\ -11 \end{bmatrix}.$$

Find G_1, G_2, z_1 and z_2 that minimize J.

Plotting $x_1, \dots, x_6$



Hence it is clear that $G_1 = \{1,2,3\}$ $z_1 = \begin{bmatrix} 10 \\ 10 \end{bmatrix}$

$G_2 = \{4,5,6\}$ $z_2 = \begin{bmatrix} -10 \\ -10 \end{bmatrix}$

More rigourous justification welcomed, but not required.

(b) Returning to general n, N and k. Does it always hold that,

$$\sum_{i \in G_j} ||x_i - z_j||^2 \leq \sum_{i \in G_j} ||x_i - z||^2,$$

for any $j \in \{1, \dots, k\}$ and any $z \in \mathbb{R}^n$? If so prove why, otherwise, explain why not or present a counter-example.

$$R = \sum_{i \in G_j} ||X_i - z||^2 = \sum ||X_i - z_j + z_j - z||^2$$

$$= \sum (X_i - z_j + z_j - z)^T (X_i - z_j + z_j - z)$$

$$= \sum ||X_i - z_j||^2 + \sum ||z_j - z||^2 + \underbrace{2 \sum (X_i - z_j)(z_j - z)}_{0}$$

$$\geq \sum ||X_i - z_j||^2 = L$$

Question 5

(a) Consider the linear dynamical system, $x_{t+1} = Ax_t$ with,

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

Find the eigenvalues of A and corresponding eigenvectors where the second coordinate of each eigenvector is 1. To simplify notation, you may use the golden ratio $\psi = \frac{1}{2}(1 + \sqrt{5})$ as well the fact that $1 - \psi = \frac{1}{2}(1 - \sqrt{5})$ and $\psi = 1/(\psi - 1)$.

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \lambda(\lambda - 1) - 1$$

$$= \lambda^2 - \lambda - 1$$

$$\lambda_{12} = \frac{1 \pm \sqrt{1+4}}{2}$$

Eigenvalues: $\lambda_1 = \psi \quad \lambda_2 = 1 - \psi$

Eigvec of $\lambda_1 = \psi$

$$\begin{bmatrix} 1-\psi & 1 \\ 1 & -\psi \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad x(1-\psi) + 1 = 0 \qquad x = \frac{1}{\psi - 1} = \psi \qquad V_1 = \begin{bmatrix} \psi \\ 1 \end{bmatrix}$$

Eig vec of $\lambda_2 = 1 - \psi$

$$\begin{bmatrix} 1-(1-\psi) & 1 \\ 1 & \psi - 1 \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad x\psi + 1 = 0 \qquad x = \frac{1}{\psi} \qquad V_2 = \begin{bmatrix} 1/\psi \\ 1 \end{bmatrix}.$$

(b) Assume now that $x_0 = [\psi\ 1]^T$. And denote the scalar q_t by,

$$q_t = [1\ 0]x_t.$$

What is the smallest (integer) t for which $q_t \geq 1000$?

observe $X_0 = V_1$ eigenvector of $\lambda_1 = \psi$

$$X_t = A X_{t-1} = \dots = A^t X_0 = \psi^t \begin{bmatrix} \psi \\ 1 \end{bmatrix} = \begin{bmatrix} \psi^{t+1} \\ \psi^t \end{bmatrix}$$

$$q_t = [1\ 0] \begin{bmatrix} \psi^{t+1} \\ \psi^t \end{bmatrix} = \psi^{t+1}$$

$$\psi^{t+1} = 1000$$

$$(t+1)\log_{10}\psi = 3$$

$$t = \frac{3}{\log_{10}\psi} - 1 = \frac{3}{0.20899} - 1 = 13.35$$

Hence first t is $t = 14$.

END OF EXAMINATION

(b) Assume now that $\lambda_1 = \lambda_2 = \ldots = \lambda_{49} = 4$. Determine,

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then $\|A\beta - V\| = \sqrt{\sum_{i=1}^{n} \left(Y_i - \beta_0 - \beta_1 X_i - \beta_2 X_i^2\right)^2}$

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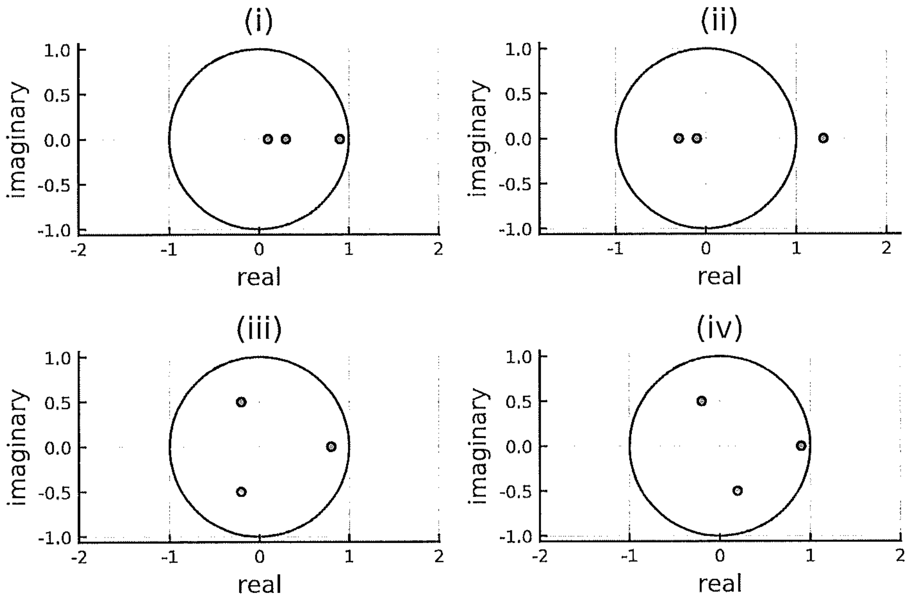

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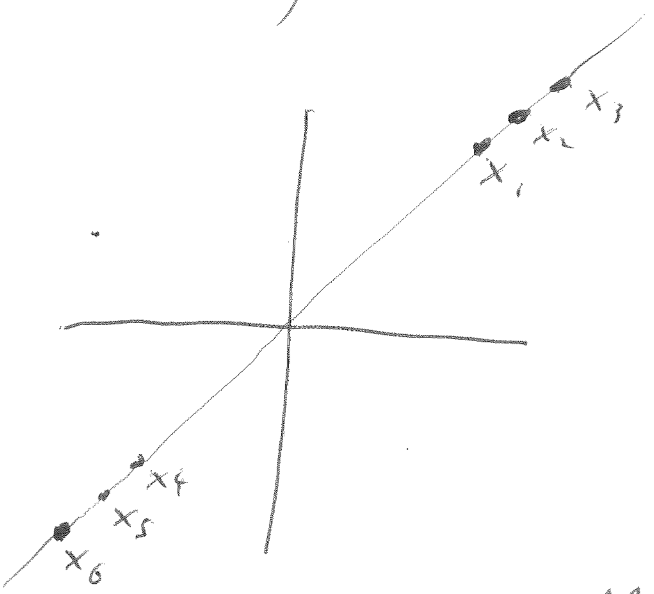

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$$R = \sum_{i \in G_j} ||X_i - Z||^2 = \sum ||X_i - Z_j + Z_j - Z||^2$$

$$= \sum (X_i - Z_j + Z_j - Z)^T (X_i - Z_j + Z_j - Z)$$

$$= \sum ||X_i - Z_j||^2 + \sum ||Z_j - Z||^2 + \underbrace{2 \sum (X_i - Z_j)(Z_j - Z)}_{0}$$

$$\geq \sum ||X_i - Z_j||^2 = L$$

Question 5

(a) Consider the linear dynamical system, $x_{t+1} = Ax_t$ with,

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

Find the eigenvalues of A and corresponding eigenvectors where the second coordinate of each eigenvector is 1. To simplify notation, you may use the golden ratio $\psi = \frac{1}{2}(1+\sqrt{5})$ as well the fact that $1-\psi = \frac{1}{2}(1-\sqrt{5})$ and $\psi = 1/(\psi - 1)$.

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$$\lambda_{12} = \frac{1 \pm \sqrt{1+4}}{2}$$

Eigenvalues: $\lambda_1 = \psi \quad \lambda_2 = 1-\psi$

Eigvec of $\lambda_1 = \psi$

$$\begin{bmatrix} 1-\psi & 1 \\ 1 & -\psi \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad x(1-\psi)+1 = 0 \qquad x = \frac{1}{\psi - 1} = \psi \qquad V_1 = \begin{bmatrix} \psi \\ 1 \end{bmatrix}$$

Eig vec of $\lambda_2 = 1-\psi$

$$\begin{bmatrix} 1-(1-\psi) & 1 \\ 1 & \psi - 1 \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad x\psi + 1 = 0 \qquad x = \frac{1}{\psi} \qquad V_2 = \begin{bmatrix} 1/\psi \\ 1 \end{bmatrix}.$$

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$$q_t = [1\ 0]x_t.$$

What is the smallest (integer) t for which $q_t \geq 1000$?

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$$X_t = A X_{t-1} = \ldots = A^t X_0 = \psi^t \begin{bmatrix} \psi \\ 1 \end{bmatrix} = \begin{bmatrix} \psi^{t+1} \\ \psi^t \end{bmatrix}$$

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