1. Consider the system of equations

$$\begin{aligned} 5x + 3y &= 2z + 4 \\ 4x &= 2y - 3x + 12 \\ x + y - 5z &= 24 \end{aligned}$$

   (a) Represent it as $Aw = b$ where $w = [x, y, z]^T$.
   (b) Solve for w numerically in Julia.
   (c) Solve manually using Gaussian elimination.
   (d) Find A^{-1} explicitly using Gaussian elimination and use it to manually obtain w via, $w = A^{-1}b$.

2. Consider two vectors $u, v \in \mathbb{R}^3$ and let $A = uv^T$. Explain why $\det(A) = 0$.

3. Consider two vectors $u, v \in \mathbb{R}^n$. Present proofs for the following:

   (a) $||u + v||^2 = ||u||^2 + 2u^T v + ||v||^2$.
   (b) $(u + v)^T (u - v) = ||u||^2 - ||v||^2$.
   (c) $|u^T v| \leq ||u|| \, ||v||$. (Cauchy-Schwarz inequality).
   (d) $||u + v|| \leq ||u|| + ||v||$. (Triangle inequality).
   (e) $||u + v||^2 + ||u - v||^2 = 2(||u||^2 + ||v||^2)$. (This is called the parallelogram law).
   (f) If $u^T v = 0$ then $||u + v||^2 = ||u||^2 + ||v||^2$. (Pythagoras theorem).

4. Find two 3×3 matrices with all non-zero entries that commute. I.e. $AB = BA$.

5. In Chapter 3 of [VMLS] there are definitions for **avg**(), **rms**(), **std**(). There is also a derivation of this identity for any vector x,

$$\mathbf{std}(x)^2 = \mathbf{rms}(x)^2 - \mathbf{avg}(x)^2.$$

   Describe the steps of the derivation of this formula in detail.

6. Do exercise 3.26 from pg. 67 of [VMLS].

7. Implement a Julia function that receives two matrices A and B and implements the product AB if they can be multiplied this way (otherwise the function throws an error). The implementation should be via dot products, linear combination of columns, linear combination of rows, or sum of outer products. A flag should indicate which of the four methods is to be used.

8. If you were give a random $n \times n$ matrix A where each $A_{i,j}$ is independently and uniformly selected from a continuous distribution, e.g. uniform(0,1), then the probability of the matrix being singular is 0. However if the entries are uniformly selected from a discrete set of numbers then there is a non zero chance to have a less than full rank matrix.

   (a) Demonstrate this claim on 3×3 matrices by generating random uniform entries for 10^6 matrices and seeing they are all non-singular. Then generate random matrices with entries from the set $\{1, 2, 3\}$ until you find a singular matrix.
   (b) Consider now 4×4 matrices with elements selected uniformly randomly from $\{\ell_1, \ell_2, \ell_3\}$ (where $\ell_i \neq \ell_j$ and $\ell_i \neq 0$). Carry out a numerical simulation experiment to make a conjecture if the distribution of the rank is affected by the values of ℓ_i or not.
   (c) If you are able to provide any explanation to your results in (b), please do so. This is optional.