1. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined via

$$f([u \quad v]^T) = \begin{bmatrix} uv^2 + e^{u+v} \\ u^2v^2 \end{bmatrix}.$$

Further, define the function $g : \mathbb{R}^2 \to \mathbb{R}^{2\times 2}$,

$$g(x) = f(x)[1 \quad 2].$$

Note here that $[1 \quad 2]$ is a row vector. Also let $z = [1 \quad -1]^T$ be a (column) vector in $\mathbb{R}^2$.

(a) Evaluate $f(z)$.
Solution:

$$f(z) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

(b) Evaluate $g(z)$.
Solution:

$$g(z) = \begin{bmatrix} 2 \\ 1 \end{bmatrix} [1 \quad 2] = \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}.$$

(c) Evaluate $||g(z)z||$.
Solution:

$$g(z)z = \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \end{bmatrix}.$$

Hence $||g(z)z|| = \sqrt{5}$.

(d) Evaluate the inner product between the two columns of $g(z)$.
Solution:

$$[2 \quad 1][4 \quad 2]^T = 8 + 2 = 10.$$

(e) Determine $\det(g(x))$ for any $x \in \mathbb{R}^2$. Explain why the answer does not depend on x.
Solution:
The matrix $g(x)$ is a rank 1 matrix obtained from an outer product hence it is singular and has 0 determinant. To see this say $f(x) = [c_1 \quad c_2]^T$. Then,

$$g(x) = \begin{bmatrix} c_1 & 2c_1 \\ c_2 & 2c_2 \end{bmatrix}.$$

Hence $\det(g(x)) = c_1 2c_2 - 2c_1c_2 = 0$.

(f) Find the Jacobian matrix $Df(u, v)$ associated with the function $f(\cdot)$.
Solution:

$$Df(u, v) = \begin{bmatrix} v^2 + e^{u+v} & 2uv + e^{u+v} \\ 2uv^2 & 2u^2v \end{bmatrix}$$

(g) Consider now the linear approximation around z at a point $x \in \mathbb{R}^2$,

$$\hat{f}(x) = f(z) + Df(z)(x - z).$$

Find a point $x^0 \in \mathbb{R}^2$ such that $\hat{f}(x^0) = 0$.
Solution:
First get,

$$Df(z) = \begin{bmatrix} 2 & -1 \\ 2 & -2 \end{bmatrix}.$$

Now,

$$\hat{f}(x^0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 & -1 \\ 2 & -2 \end{bmatrix} \left(x^0 - \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right) = 0.$$

or,

$$\begin{bmatrix} 2 & -1 \\ 2 & -2 \end{bmatrix} x^0 = - \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 & -1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

or,

$$\begin{bmatrix} 2 & -1 \\ 2 & -2 \end{bmatrix} x^0 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

After an elimination step,

$$\begin{bmatrix} 2 & -1 \\ 0 & -1 \end{bmatrix} x^0 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Hence setting $x^0 = [u \;\; v]^T$ we have $v = -2$ and with back substation in $2u - 1(-2) = 1$ we have that $u = -1/2$. That is,

$$x^0 = \begin{bmatrix} -1/2 \\ -2 \end{bmatrix}.$$

2. Let A and B be two upper triangular $n \times n$ matrices. That is for $i > j$, $A_{i,j} = 0$ and $B_{i,j} = 0$. Consider now the unit vector $e_n \in \mathbb{R}^n$ with 0 entries everywhere except the last entry which is 1. Determine the value of $e_n^T A B e_n$.
Solution:
First observe that for a square $n \times n$ matrix C, the scalar $e_n^T C e_n$ is the entry C_{nn} (the bottom right entry). This is because multiplying from the right by e_n picks off the n'th column of C and multiplying from the left by e_n^T picks off the n'th row of C. So now the problem is to evaluate the entry (n, n) or bottom right entry of the product $C = AB$.

Now lets consider the bottom right entry (n, n) of the matrix $C = AB$.

$$C_{nn} = \sum_{k=1}^{n} A_{nk} B_{kn} = \sum_{k=1}^{n-1} A_{nk} B_{kj} + A_{nn} B_{nn} = \sum_{k=1}^{n-1} 0 \, B_{kj} + A_{nn} B_{nn}$$

Hence we see that $C_{nn} = A_{nn} B_{nn}$ and thus

$$e_n^T A B e_n = A_{nn} B_{nn}.$$

That is it is the product of the two bottom right entries of the matrices A and B.

Note that you can get this in other ways, for example you can consider $e_n^T A B e_n$ as $(e_n^T A)(B e_n)$.

3. Let $u, v \in \mathbb{R}^n$. Use the definition of the 2-norm $||\cdot||$ to prove,

$$\frac{1}{2}||u+v||^2 + \frac{1}{2}||u-v||^2 - ||u||^2 - ||v||^2 = 0.$$

Solution:
This is sometimes called the "Parallelogram law":

$$\begin{aligned}
&\frac{1}{2}||u+v||^2 + \frac{1}{2}||u-v||^2 - ||u||^2 - ||v||^2 \\
&= \frac{1}{2}(u+v)^T(u+v) + \frac{1}{2}(u-v)^T(u-v) - u^Tu - v^Tv \\
&= \frac{1}{2}u^Tu + u^Tv + \frac{1}{2}v^Tv + \frac{1}{2}u^Tu - u^Tv + \frac{1}{2}v^Tv - u^Tu - v^Tv \\
&= 0.
\end{aligned}$$