Please make sure to follow the hand-in instructions. Also, please present your answers in order, showing the working for each answer. Answering yes/no is not enough. You should rather present an argument or derivation of your answer.

Consider the vectors $v_1 = [-1\ \ 0\ \ -1]^T$, $v_2 = [0\ \ -1\ \ 1]^T$, and $v_3 = [0\ \ 1\ \ 1]^T$. Let the 3×3 matrix A have columns v_1, v_2, and v_3. That is $A = [v_1\ \ v_2\ \ v_3]$.

1. Do the vectors v_1, v_2, and v_3 constitute a set of orthogonal vectors?

   Solution:
   No. For example $v_1^T v_2 = -1$ and not 0.

2. Are the vectors v_2 and v_3 orthogonal?

   Solution:
   Yes. $v_2^T v_3 = 0 \times 0 + (-1) \times 1 + 1 \times 1 = 0$.

3. Execute the Gram-Schmidt procedure on the vectors v_1, v_2, and v_3 in this order. Present the output as columns of an orthogonal matrix Q.

   Solution:

   $\tilde{q}_1 = v_1$.
   $\tilde{q}_1 \neq 0$ so not stopping.
   $||\tilde{q}_1|| = \sqrt{2}$ hence
   $q_1 = [-1/\sqrt{2} \quad 0 \quad -1/\sqrt{2}]^T$.
   $\tilde{q}_2 = v_2 - (q_1^T v_2) q_1 = [0\ \ -1\ \ 1]^T - (-1/\sqrt{2})[-1/\sqrt{2} \quad 0 \quad -1/\sqrt{2}]^T = [-1/2 \quad -1 \quad 1/2]^T$.
   $\tilde{q}_2 \neq 0$ so not stopping.
   $||\tilde{q}_2|| = \sqrt{3/2}$ hence
   $q_2 = [-1/\sqrt{6} \quad -\sqrt{2/3} \quad 1/\sqrt{6}]^T$.

   $$\begin{aligned}\tilde{q}_3 &= v_3 - (q_1^T v_3) q_1 - (q_2^T v_3) q_2 \\ &= [0\ \ 1\ \ 1]^T - (-1/\sqrt{2})[-1/\sqrt{2} \quad 0 \quad -1/\sqrt{2}]^T - (-1/\sqrt{6})[-1/\sqrt{6} \quad -\sqrt{2/3} \quad 1/\sqrt{6}]^T \\ &= [-2/3 \quad 2/3 \quad 2/3]^T.\end{aligned}$$

   $\tilde{q}_3 \neq 0$ so not stopping.
   $||\tilde{q}_3|| = 2/\sqrt{3}$ hence
   $q_3 = [-1/\sqrt{3} \quad -1/\sqrt{3} \quad 1/\sqrt{3}]^T$.

   Hence
   $$Q = \begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{6} & -1/\sqrt{3} \\ 0 & -\sqrt{2/3} & 1/\sqrt{3} \\ -1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \end{bmatrix}.$$

4. Determine Q^{-1}.

   Solution:
   Because Q is an orthogonal matrix, $Q^{-1} = Q^T$.

5. What is the nullspace of the matrix A?

   Solution:
   Since we found via Gram-Schmidt that the vectors are independent, the nullspace has only the zero vector. That is the null space is $\{[0\ \ 0\ \ 0]^T\}$.

6. What is the rank of the matrix A?

   Solution:
   Since it is a square matrix and has all independent columns, the rank is 3.

7. Are the vectors v_1, v_2, and v_3 a basis for $\mathbb{R}^3$?

   Solution:
   Yes, the vectors span $\mathbb{R}^3$ and are linearly independent, so they are a basis.

8. Now execute the Gram-Schmidt procedure on the vectors v_3, v_2, and v_1 in the order $(3 \to 2 \to 1)$. Present the output as the columns of an orthogonal matrix M.

   Solution:

   $\tilde{q}_1 = v_3$.
   $\tilde{q}_1 \neq 0$ so not stopping.
   $||\tilde{q}_1|| = \sqrt{2}$ hence
   $q_1 = [0 \quad 1/\sqrt{2} \quad 1/\sqrt{2}]^T$.
   $\tilde{q}_2 = v_2 - (q_1^T v_2)q_1 = [0 \quad -1 \quad 1]^T - 0[0 \quad 1/\sqrt{2} \quad 1/\sqrt{2}]^T = [0 \quad -1 \quad 1]^T$.
   $||\tilde{q}_2|| = \sqrt{2}$ hence
   $q_2 = [0 \quad -1/\sqrt{2} \quad 1/\sqrt{2}]^T$.

   $$\begin{aligned}\tilde{q}_3 &= v_3 - (q_1^T v_3)q_1 - (q_2^T v_3)q_2 \\ &= [0 \;\; 1 \;\; 1]^T - (-1/\sqrt{2})[0 \quad 1/\sqrt{2} \quad 1/\sqrt{2}]^T - (-1/\sqrt{2})[0 \quad -1/\sqrt{2} \quad 1/\sqrt{2}]^T \\ &= [-1 \;\; 0 \;\; 0]^T.\end{aligned}$$

   $||\tilde{q}_3|| = 1$ hence
   $q_3 = [-1 \;\; 0 \;\; 0]^T$.

   Hence
   $$M = \begin{bmatrix} 0 & 0 & -1 \\ 1\sqrt{2} & -1\sqrt{2} & 0 \\ 1\sqrt{2} & 1\sqrt{2} & 0 \end{bmatrix}.$$

9. Define the matrix $B = QM$. Is B an orthogonal matrix?

   Solution:
   Yes it is orthogonal. To see this check that $B^T B = I$:
   $$B^T B = (QM)^T QM = M^T Q^T QM = M^T IM = I.$$

10. Consider now the 9×9 block matrix,
    $$G = \begin{bmatrix} Q & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & Q \end{bmatrix},$$
    where each 0 is a 3×3 matrix of zero values. What is the inverse matrix of G?

    Solution:
    The matrix G is also an orthogonal matrix and its is inverse is G^T.
    $$G^T = \begin{bmatrix} Q^T & 0 & 0 \\ 0 & M^T & 0 \\ 0 & 0 & Q^T \end{bmatrix}.$$