Please make sure to follow the hand-in instructions. Also, please present your answers in order, showing the working for each answer. Answering yes/no is not enough. You should rather present an argument or derivation of your answer. **Tip:** Do NOT waste time on excessive computations because the quiz can be solved without requiring big computations.

1. Consider the vector $u = [1\ \ 2]^T$ and the vector $v = [1\ \ 10]^T$. Let the matrix A be the outer product $A = u\,v^T$. Determine the eigenvalues of A.
Solution:
The problem can be solved like the next one, but a straight forward solution is via the characteristic polynomial,

$$\det\left(\begin{bmatrix} 1-\lambda & 10 \\ 2 & 20-\lambda \end{bmatrix}\right) = (1-\lambda)(20-\lambda) - 20 = \lambda^2 - 21\lambda = \lambda(\lambda - 21).$$

Hence by equating to 0 we see that the eigenvalues are $\lambda = 0$ and $\lambda = 21$.

2. Now set $u = [1\ \ 2\ \ 3\ \ 4\ \ 5]^T$ and $v = [1\ \ 10\ \ 10^2\ \ 10^3\ \ 10^4]^T$. With these values determine the eigenvalues of $A = u\,v^T$. Hint: Remember that the sum of the eigenvalues is the trace.
Solution:
In this case computing (and solving the roots of) the characteristic polynomial is more difficult. However we note that A is a matrix with a rank of one and hence all of its eigenvalues except for a one, are 0. To see this (no need for this proof to get marks), consider the eigenvalue/eigenvector equation

$$(A - \lambda I)x = 0.$$

By setting $\lambda = 0$ we see that any non-zero vector x in the null-space of A is an eigenvector. Now since the rank$(A) = 1$, the null-space is of dimension $5 - 1 = 4$ and hence is spanned by 4 distinct eigenvectors, each corresponding to the eigenvalue $\lambda = 0$.

The question is now about the fifth eigenvalue. Here we can use the fact that trace(A), the sum of the diagonal elements of A is also the sum of the eigenvalues. Since 4 of them are 0, the trace equals the fifth eigenvalue. Now the trace of an outer product (matrix) is simply the inner product of the vectors that make up the matrix, so the fifth eigenvalue is,

$$\lambda = u^T v = 54,321.$$

3. Let B be a 3×2 matrix with rank$(B) = 2$, and set the vector $c = [0\ \ 1\ \ 1]^T$. Assume that,

$$B(B^T B)^{-1} B^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 1/2 \\ 0 & 1/2 & 1/2 \end{bmatrix}.$$

Let x^* be the value of x that minimizes $||Bx - c||$. Determine the value of Bx^* and the value of $||Bx^* - c||$.
Solution:
Here $P = B(B^T B)^{-1} B^T$ is a projection matrix associated with this least squares problem. The x^* that minimizes $||Bx - c||$ is $x^* = B^\dagger c$ where $B^\dagger = (B^T B)^{-1} B^T$. Now

$$Bx^* = BB^\dagger c = Pc.$$

Hence,

$$Bx^* = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 1/2 \\ 0 & 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = c.$$

That is, for this specific c we have that c is in the column space of B and the least squares problem is "solved exactly". In this case, $||Bx^* - c|| = 0$.

4. Continuing with the same B, prove that $B^T B$ is a positive definite matrix.
Solution:
Take $x \neq 0$ and consider,

$$x^T B^T B x = (Bx)^T B x = ||Bx||^2.$$

We know that B is a skinny full-rank matrix, that is its columns are linearly independent. Hence for $x \neq 0$, the vector Bx is also not 0. Hence the norm $||Bx||$ is strictly positive and hence, $x^T B^T B x > 0$ showing that $B^T B$ is positive definite.

5. What are the 3 eigenvalues of $B(B^T B)^{-1} B^T$? Why?
Solution:
The matrix $P = B(B^T B)^{-1} B^T$ is a projection matrix on a subspace of dimension 2. This means that non-zero vectors that are linear combinations of the columns of B are eigenvectors of P with an eigenvalue of 1. To see this formally multiply P by each column or all-together,

$$PB = B(B^T B)^{-1} B^T B = B = 1B.$$

Further, vectors that are orthogonal to the column space of B are projected onto the 0 vector and hence have an eigenvalue of 0. To see this formally, any such vector, say w, satisfies. $w^T B = 0$ (here 0 is a row vector of length 3). Alternatively, $B^T w = 0$ (here 0 is a column vector of length 3). Hence,

$$Pw = B(B^T B)^{-1} B^T w = 0 = 0w.$$

(Note that the first 0 in the above equation is a vector and the second is a scalar). Hence 0 is an eigenvalue and $w \neq 0$ is an eigenvector.

Because the rank of B is 2 there are two independent eigenvectors with eigenvalue 1 and a single (independent) eigenvector with eigenvalue 0. Hence the eigenvalues of P are 0, 1, and 1.

6. Consider the sequence $x(0), x(1), x(2), \ldots$ of vectors in $\mathbb{R}^3$ with

$$x(k+1) = B(B^T B)^{-1} B^T x(k) - \frac{1}{2} x(k),$$

for $k = 1, 2, \ldots$. Argue about the value of the limit,

$$\lim_{k \to \infty} ||x(k)||.$$

Does it depend on $x(0)$? Or is it the same limit for all $x(0)$? What is the limit?
Solution:
The recursion can be written as,

$$x(k+1) = Rx(k),$$

where $R = P - \frac{1}{2} I$. This means that the eigenvalues of R are those of P minus 1/2. They are then $-\frac{1}{2}$, $\frac{1}{2}$, and $\frac{1}{2}$. This makes R a *stable* matrix which means that for any initial condition $x(k) \to 0$. Hence the norm converges to 0. One argument for this (not needed to get marks for the points) is to diagonalize Q as $R = Q \Lambda Q^T$ where Q is an orthogonal matrix of eigenvectors (exists here R is symmetric) and Λ has the eigenvalues on the diagonal. Then,

$$x(k) = R^k x(0) = Q \Lambda^k Q^T x(0).$$

This then converges to the 0 vector because each element of the diagonal of Λ^k is an eigenvalue in the range $(-1, 1)$ raised to the power k.