STAT2201, Semester 1 2016

Solution for Quiz #2 (versions "A" and "B") – Solution amended: May 10

Let M be your month of birth, i.e. $M \in \{1, 2, \ldots, 12\}$. $M =$ ______________.

Question 1: Assume you are faced with a random sample Y_1, Y_2, Y_3, Y_4 coming from a normal population with mean 3 and variance 8^2. Denote the sample mean by $\overline{Y}$.

(aA) $\Pr\big(\ Y_1\ \leq \frac{M}{3}\big) =$ ______________
(aB) $\Pr\big(\ Y_1\ > \frac{M}{3}\big) =$ ______________
(bA) $\Pr\big(\ \overline{Y}\ \leq \frac{M}{3}\big) =$ ______________
(bB) $\Pr\big(\ \overline{Y}\ > \frac{M}{3}\big) =$ ______________

Solution: Here Y_1 is distributed as $N(3,\, 8^2)$ and the sample mean $\overline{Y}$ is distributed $N(3, \frac{8^2}{4})$ because it is a sample based on 4. Hence the standard deviation of a single observation, Y_1 is 8 and the standard deviation of the sample mean, $\overline{Y}$ is $\frac{8}{\sqrt{4}} = 4$.

Now the problem is just computation of probabilities of normal random variables. So by subtracting the mean and dividing by the appropriate standard deviation, the problem is transformed to computation of probabilities of standard normal random variable Z:

$$\Pr\Big(\ Y_1\ \leq \frac{M}{3}\Big) = \Pr\Big(\ Z\ \leq \frac{M/3 - 3}{8}\Big)$$
$$\Pr\Big(\ Y_1\ > \frac{M}{3}\Big) = \Pr\Big(\ Z\ > \frac{M/3 - 3}{8}\Big)$$
$$\Pr\Big(\ \overline{Y}\ \leq \frac{M}{3}\Big) = \Pr\Big(\ Z\ \leq \frac{M/3 - 3}{4}\Big)$$
$$\Pr\Big(\ \overline{Y}\ > \frac{M}{3}\Big) = \Pr\Big(\ Z\ > \frac{M/3 - 3}{4}\Big)$$

Plugging in values we obtain:

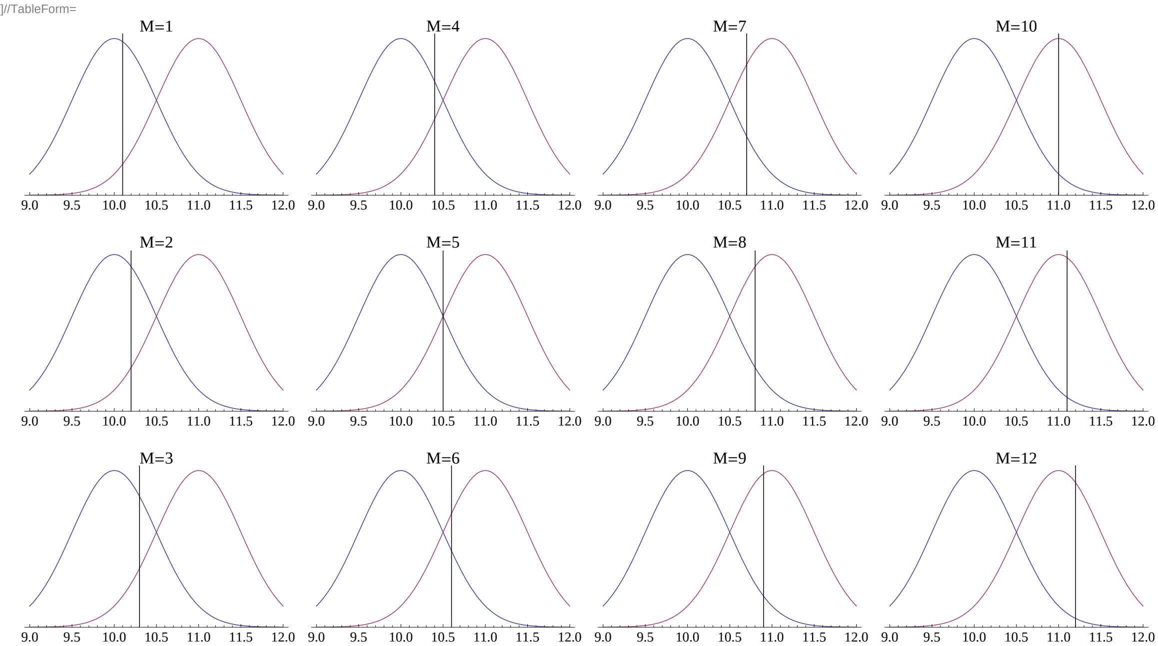
	$M=1$	$M=2$	$M=3$	$M=4$	$M=5$	$M=6$	$M=7$	$M=8$	$M=9$	$M=10$	$M=11$	$M=12$
a_A	0.369441	0.385271	0.401294	0.417484	0.433816	0.450262	0.466793	0.483382	0.5	0.516618	0.533207	0.549738
a_B	0.630559	0.614729	0.598706	0.582516	0.566184	0.549738	0.533207	0.516618	0.5	0.483382	0.466793	0.450262
b_A	0.252493	0.279834	0.308538	0.338461	0.369441	0.401294	0.433816	0.466793	0.5	0.533207	0.566184	0.598706
b_B	0.747507	0.720166	0.691462	0.661539	0.630559	0.598706	0.566184	0.533207	0.5	0.466793	0.433816	0.401294

Question 2A: There are two types of bricks. Both types are normally distributed with unit variance. *Regular* bricks have a mean weight of $\mu_0 = 10$kg. *Deluxe* bricks have a mean weight of $\mu_A = 11$kg. You collect a random a sample of 4 bricks known to be either all regular or all deluxe, but you are not sure which.
To make a decision, you design the following hypothesis test: You set $H_0: \ \mu = \mu_0$ and $H_A: \ \mu = \mu_A$. As the test statistic you take the sample mean, $\overline{Y}$ and compare it to a threshold τ. If $\overline{Y} < \tau$ you retain H_0 and otherwise you reject H_0 and accept H_A. Set now $\tau = 10 + \frac{M}{10}$.

(a) Draw a sketch of the hypothesis test, showing the distribution of $\overline{Y}$ under H_0 and H_A and indicating the location of τ.
(b) Type I error = ____________________
(c) Type II error = ____________________

Solution: The standard deviation of the sample mean, $\overline{Y}$ is $1/\sqrt{4} = 0.5$. Here are plots of the distributions of $\overline{Y}$ with H_0 on the left and H_A on the right, for all values of the threshold τ:

Out[59]//TableForm=



The Type I error is the area under the blue curve to the right of the threshold (vertical) line. The type II error is the area under the purple curve to the left of the threshold (vertical) line.

In calculating the probabilities of the type I and type II errors:

$$\Pr(\text{Type I error}) = \Pr(\overline{Y}_{\mu=10} > \tau) = \Pr(Z > \frac{\tau - 10}{0.5}) = \Pr(Z > \frac{M}{5})$$
$$\Pr(\text{Type I error}) = \Pr(\overline{Y}_{\mu=11} < \tau) = \Pr(Z < \frac{\tau - 11}{0.5}) = \Pr(Z < \frac{M}{5} - 2)$$

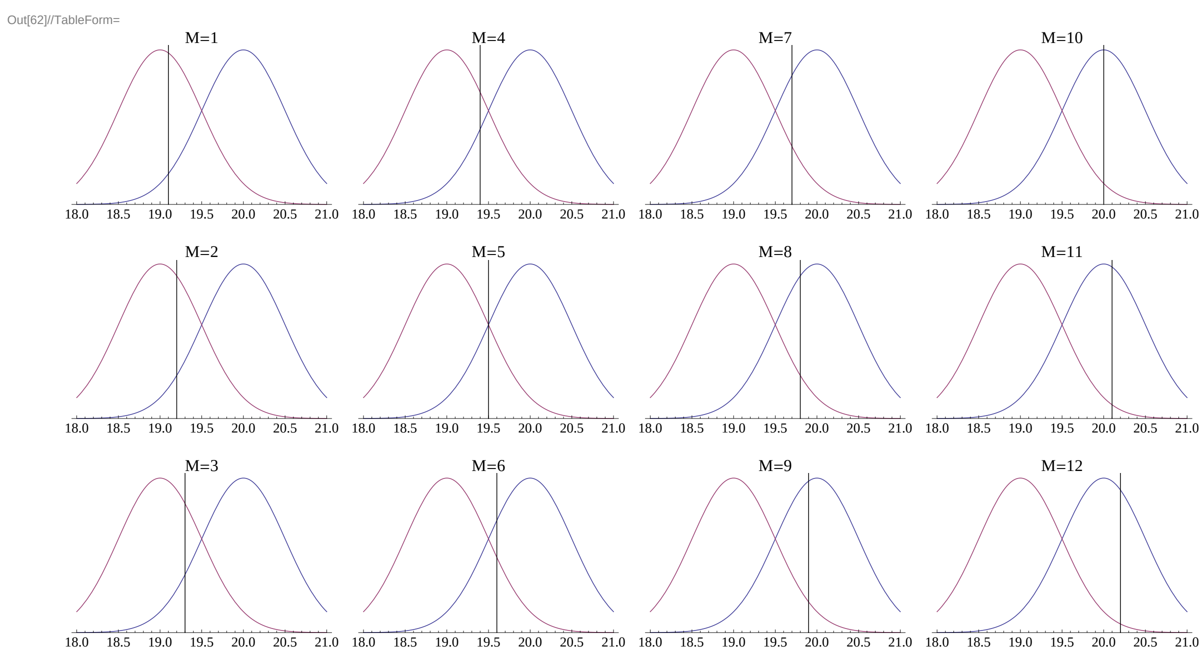
Plugging in values we obtain:

	$M=1$	$M=2$	$M=3$	$M=4$	$M=5$	$M=6$	$M=7$	$M=8$	$M=9$	$M=10$	$M=11$	$M=12$
(a)	0.4207	0.3446	0.2743	0.2119	0.1587	0.1151	0.08076	0.05480	0.03593	0.02275	0.01390	0.008198
(b)	0.03593	0.05480	0.08076	0.1151	0.1587	0.2119	0.2743	0.3446	0.4207	0.5000	0.5793	0.6554

Question 2B: (this is a slightly different version) There are two types of bricks. Both types are normally distributed with unit variance. *Regular* bricks have a mean weight of $\mu_0 = 20$kg. *Deluxe* bricks have a mean weight of $\mu_A = 19$kg. You collect a random a sample of 4 bricks known to be either all regular or all deluxe, but you are not sure which.
To make a decision, you design the following hypothesis test: You set $H_0: \ \mu = \mu_0$ and $H_A: \ \mu = \mu_A$. As the test statistic you take the sample mean, $\overline{Y}$ and compare it to a threshold τ. If $\overline{Y} > \tau$ you retain H_0 and otherwise you reject H_0 and accept H_A. Set now $\tau = 19 + \frac{M}{10}$.

(a) Draw a sketch of the hypothesis test, showing the distribution of $\overline{Y}$ under H_0 and H_A and indicating the location of τ.
(b) Type I error = ____________________
(c) Type II error = ____________________

Solution: As in the previous version, the standard deviation of the sample mean, $\overline{Y}$ is $1/\sqrt{4} = 0.5$. Here are plots of the distributions of $\overline{Y}$ with H_0 on the **right** and H_A on the **left**, for all values of the threshold τ:



The Type I error is the area under the blue curve to the **left** of the threshold (vertical) line. The type II error is the area under the purple curve to the **right** of the threshold (vertical) line.

In calculating the probabilities of the type I and type II errors:

$$\Pr(\text{Type I error}) = \Pr(\overline{Y}_{\mu=20} < \tau) = \Pr(Z < \frac{\tau - 20}{0.5}) = \Pr(Z < \frac{M}{5} - 2)$$
$$\Pr(\text{Type I error}) = \Pr(\overline{Y}_{\mu=19} > \tau) = \Pr(Z > \frac{\tau - 19}{0.5}) = \Pr(Z > \frac{M}{5})$$

Notice that these values are exactly as for version A above, only that the type I and type II error are reversed. Namely, plugging in values:

	$M=1$	$M=2$	$M=3$	$M=4$	$M=5$	$M=6$	$M=7$	$M=8$	$M=9$	$M=10$	$M=11$	$M=12$
(a)	0.03593	0.05480	0.08076	0.1151	0.1587	0.2119	0.2743	0.3446	0.4207	0.5000	0.5793	0.6554
(b)	0.4207	0.3446	0.2743	0.2119	0.1587	0.1151	0.08076	0.05480	0.03593	0.02275	0.01390	0.008198

NORMAL CUMULATIVE DISTRIBUTION FUNCTION: $\Pr(Z \leq x)$

x	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7703	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998
3.5	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998
3.6	0.9998	0.9998	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.7	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.8	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.9	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000