UQ, STAT2201, 2017,
Lecture 2, Unit 2,
Probability and Monte Carlo.

I'm willing to bet that there are two people in the room with the same birthday!

We'll revisit this "birthday problem" a few times during the lecture.

Sample Space, Outcomes and Events

An experiment that can result in different outcomes, even though it is repeated in the same manner every time, is called a **random experiment**.

The set of all possible outcomes of a random experiment is called the **sample space** of the experiment, and is denoted as Ω.
(In [MonRun2014] denoted as S).

A sample space is **discrete** if it consists of a finite or countable infinite set of outcomes.

A sample space is **continuous** if it contains an interval (either finite or infinite) of real numbers, vectors or similar objects.

Example 2-1 **Camera Flash** Consider an experiment that selects a cell phone camera and records the recycle time of a flash (the time taken to ready the camera for another flash). The possible values for this time depend on the resolution of the timer and on the minimum and maximum recycle times. However, because the time is positive it is convenient to define the sample space as simply the positive real line

$$S = R^+ = \{x \mid x > 0\}$$

If it is known that all recycle times are between 1.5 and 5 seconds, the sample space can be

$$S = \{x \mid 1.5 < x < 5\}$$

If the objective of the analysis is to consider only whether the recycle time is low, medium, or high, the sample space can be taken to be the set of three outcomes

$$S = \{low, medium, high\}$$

If the objective is only to evaluate whether or not a particular camera conforms to a minimum recycle time specification, the sample space can be simplified to a set of two outcomes

$$S = \{yes, no\}$$

that indicates whether or not the camera conforms.

An **event** is a subset of the sample space of a random experiment.

The **union** of two events is the event that consists of all outcomes that are contained in either of the two events or both. We denote the union as $E_1 \cup E_2$.

The **intersection** of two events is the event that consists of all outcomes that are contained in both of the two events. We denote the intersection as $E_1 \cap E_2$.

The **complement** of an event in a sample space is the set of outcomes in the sample space that are not in the event.

We denote the complement of the event E as $\overline{E}$. The notation E^C is also used in other literature to denote the complement.

Note that $E \cup \overline{E} = \Omega$.

Union and intersection are commutative operations:
$A \cap B = B \cap A \qquad \text{and} \qquad A \cup B = B \cup A.$

Two events, denoted E_1 and E_2 are **mutually exclusive** if:
$E_1 \cap E_2 = \varnothing$ where $\varnothing$ is called the **empty set** or **null event**.

A collection of events, $E_1, E_2, \ldots, E_k$ is said to be **mutually exclusive** if for all pairs,

$$E_i \cap E_j = \varnothing.$$

The distributive law for set operations:

$$(A \cup B) \cap C = (A \cap C) \cup (B \cap C),$$
$$(A \cap B) \cup C = (A \cup C) \cap (B \cup C).$$

DeMorgan's laws:

$$(A \cup B)^c = A^c \cap B^c \qquad \text{and} \qquad (A \cap B)^c = A^c \cup B^c.$$

See in Julia, that $!(A \ || \ B) \ == \ !A \ \&\& \ !B$.

Example 2-7

As in Example 2-1, camera recycle times might use the sample space $S = R^{+}$, the set of positive real numbers. Let

$$E_1 = \{x \mid 10 \le x < 12\} \quad \text{and} \quad E_2 = \{x \mid 11 < x < 15\}$$

Then,

$$E_1 \cup E_2 = \{x \mid 10 \le x < 15\}$$

and

$$E_1 \cap E_2 = \{x \mid 11 < x < 12\}$$

Also,

$$E_1' = \{x \mid x < 10 \quad \text{or} \quad 12 \le x\}$$

and

$$E_1' \cap E_2 = \{x \mid 12 \le x < 15\}$$

Probability

Probability is used to quantify the likelihood, or chance, that an outcome of a random experiment will occur.

Whenever a sample space consists of a finite number N of possible outcomes, each **equally likely**, the probability of each outcome is $1/N$.

For a discrete sample space, the **probability of an event** E, denoted as $P(E)$, equals the sum of the probabilities of the outcomes in E.

If Ω is the sample space and E is any event in a random experiment,

(1) $P(\Omega) = 1$.

(2) $0 \leq P(E) \leq 1$.

(3) For two events E_1 and E_2 with $E_1 \cap E_2 = \varnothing$ (disjoint),
$$P(E_1 \cup E_2) = P(E_1) + P(E_2).$$

(4) $P(E^c) = 1 - P(E)$.

(5) $P(\emptyset) = 0$.

Back to the birthday problem:

$$n = \text{Number of students in class.}$$

$$E = \{\text{Two or more shared birthdays}\}.$$

$$P(E) = ?$$

$$P(E) = 1 - P(E^c).$$

$$E^c = \{\text{No one has the same birthday}\}.$$

Take $n = 3$:

$$P(E^c) = P(\text{No same birthday})$$
$$= \frac{\text{number outcomes without same birthday}}{\text{number of possible birthday outcomes}}$$
$$= \frac{365 \cdot 364 \cdot 363}{365^3}.$$

Take general $n \leq 365$:

$$
\begin{aligned}
P(E^c) &= P(\text{No same birthday}) \\
&= \frac{\text{number outcomes without same birthday}}{\text{number of possible birthday outcomes}} \\
&= \frac{365 \cdot 364 \cdot \ldots \cdot (365 - n + 1)}{365^n} \\
&= \frac{365!/(365 - n)!}{365^n}
\end{aligned}
$$

Try it in Julia.

Probabilities of Unions

The probability of event A or event B occurring is,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

If A and B are mutually exclusive events,

$$P(A \cup B) = P(A) + P(B).$$

For a collection of **mutually exclusive events**,

$$P(E_1 \cup E_2 \cup \cdots \cup E_k) = P(E_1) + P(E_2) + \ldots P(E_k).$$

Conditional Probability and Independence

The probability of an event B under the knowledge that the outcome will be in event A is denoted $P(B \mid A)$ and is called the **conditional probability** of B given A.

The **conditional probability** of an event B given an event A, denoted as $P(B \mid A)$, is

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)} \quad \text{for} \quad P(A) > 0.$$

The **multiplication rule** for probabilities is:
$P(A \cap B) = P(B \mid A)P(A) = P(A \mid B)P(B)$.

For an event B and a collection of mutual exclusive events, $E_1, E_2, \ldots, E_k$ where their union is Ω. The **law of total probability** yields,

$$
\begin{aligned}
P(B) &= P(B \cap E_1) + P(B \cap E_2) + \cdots + P(B \cap E_k) \\
&= P(B \mid E_1)P(E_1) + P(B \mid E_2)P(E_2) + \cdots + P(B \mid E_k)P(E_k).
\end{aligned}
$$

Two events are **independent** if any one of the following equivalent statements is true:

(1) $P(A \mid B) = P(A)$.

(2) $P(B \mid A) = P(B)$.

(3) $P(A \cap B) = P(A)P(B)$.

Observe that **independent** events and **mutually exclusive** events, are completely different concepts. Don't confuse these concepts.

For **multiple events** $E_1, E_2, \ldots, E_n$ are independent if and only if for any subset of these events

$$P(E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_k}) = P(E_{i_1})\ P(E_{i_2})\ \ldots\ P(E_{i_k}).$$

Example 2-28 **Semiconductor Failures** Continuing with semiconductor manufacturing, assume the following probabilities for product failure subject to levels of contamination in manufacturing:

Probability of Failure	Level of Contamination
0.10	High
0.01	Medium
0.001	Low

In a particular production run, 20% of the chips are subjected to high levels of contamination, 30% to medium levels of contamination, and 50% to low levels of contamination. What is the probability that a product using one of these chips fails? Let

- H denote the event that a chip is exposed to high levels of contamination
- M denote the event that a chip is exposed to medium levels of contamination
- L denote the event that a chip is exposed to low levels of contamination

Then,

$$P(F) = P(F \mid H)P(H) + P(F \mid M)P(M) + P(F \mid L)P(L)$$
$$= 0.10(0.20) + 0.01(0.30) + 0.001(0.50) = 0.0235$$

The calculations are conveniently organized with the tree diagram in Fig. 2-17.

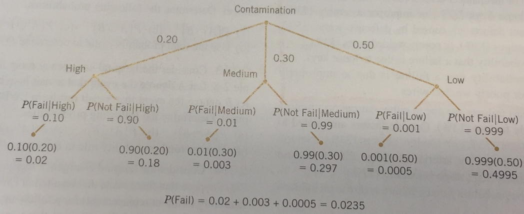

FIGURE 2-17 Tree diagram for Example 2-28.

Monte Carlo

Computer simulation of random experiments is called **Monte Carlo** and is typically carried out by setting the seed to either a reproducible value or an arbitrary value such as system time.

Random experiments may be replicated on a computer using Monte Carlo simulation.

A **pseudorandom sequence** is a sequence of numbers $U_1, U_2, \ldots$ with each number, U_k depending on the previous numbers $U_{k-1}, U_{k-2}, \ldots, U_1$ through a well defined functional relationship and similarly U_1 depending on the **seed** $\tilde{U}_0$. Hence for any seed, $\tilde{U}_0$, the resulting sequence $U_1, U_2, \ldots$ is fully defined and repeatable. A pseudorandom often lives within a discrete domain as $\{0, 1, \ldots, 2^{64} - 1\}$.

It can then be **normalised** to floating point numbers with,

$$R_k = \frac{U_k}{2^{64} - 1}.$$

A good pseudorandom sequence has the following attributes among others:

- It is quick and easy to compute the next element in the sequence.
- The sequence of numbers $R_1, R_2, \dots$ resembles properties as an i.i.d. sequence of uniform(0,1) random variables (i.i.d. is defined in Unit 4).

Exploring The Birthday Problem