Question 1. A Digital Scale

A digital scale that provides weights to the nearest gram is used.

(a) What is the sample space for this experiment?

a. The positive integers $\mathbb{Z}^+$

Let A denote the event that a weight exceeds 12 grams, let B denote the event that a weight is less than or equal to 14 grams, and let C denote the event that a weight is greater than or equal to 9 grams and less than 14 grams. Describe the events:

(b) $A \cup B$

As this would be the event of either A, B or both happening this means the weight will exceed 12 grams or be less than or equal to 14 grams. Therefore the event is any weight in the sample space, Ω

(c) $A \cap B$

This is the event of A and B happening at the same time. This means the weight will exceed 12 grams but be less than or equal to 14 grams. Therefore this is the event of the weight being either 13 or 14 grams

(d) $\overline{A}$

This event is the compliment of A or NOT A. So this event is that the weight will be less than or equal to 12 grams.

(e) $A \cup (B \cap C)$

$B \cap C$ is the event that a weight is less than or equal to 14 grams and that it is greater than or equal to 9 grams and less than 14 grams, which means event that it is simple event C. Rewriting we get $A \cup (B \cap C) = A \cup C$, which is the event that the weight is greater than 9 grams.

(f) $\overline{(A \cup C)}$

$A \cup C$ is the event that the weight is greater than 12 grams or greater than or equal to 9 grams and less than 14 grams, so overall greater than or equal to 9 grams. As we are looking for the compliment we need the disjoint event to this, which is that the weight is less than 9 grams.

(g) $A \cap B \cap C$

This event is the intersection of all three events, which is the point all of the events overlap. The overlap between B and C is simple the event C, so we can rewrite this as $A \cap C$. There is only one weight that is both greater than 12 grams and greater or equal to 9 grams and less than 14 grams. So the event is that the weight is 13 grams.

(h) $\overline{B} \cap C$

The event NOT B is that the weight is greater than 14. This is a disjoint event to C (i.e. no intersection) so the intersection is the empty set, $\emptyset$.

Question 2 - Transmitting Bits

Four bits are transmitted over a digital communications channel. Each bit is either distorted or received without distortion. Let A_i denote the event that the ith bit is distorted, $i = 1, \ldots, 4.$

(a) Describe the sample space for this experiment.

Here the sample space is all the combinations of four bits either distorted or not distorted, i.e.
$\Omega = \{0,1\}^4 = \{0000, 0001, 0010, 0011, 0100, 0101, 0110, 0111, 1000, 1001, 1010, 1011, 1100, 1101, 1110, 1111$

(b) Are the A_i's mutually exclusive?

As can be seen above more than one bit can be distorted and they can happen in any order, therefore they are not mutually exclusive.

Describe the outcomes in each of the following events:

(c) A_1

The first bit is distorted $\{1000, 1001, 1010, 1011, 1100, 1101, 1110, 1111\}$

(d) $\overline{A_1}$

This is the compliment to the previous event. This event is the event that the first bit is not distorted $\{0000, 0001, 0010, 0011, 0100, 0101, 0110, 0111\}$

(e) $A_1 \cap A_2 \cap A_3 \cap A_4$

This is the event that all bits are distorted $\{1111\}$

(f) $(A_1 \cap A_2) \cup (A_3 \cap A_4)$

This is the event that the first two bits are distorted or the last two bits are distorted $\{1100, 1101, 1110, 1111, 0111, 1011, 0011\}$. Note, all bits can be distorted in this event.

Question 3 - Basic Simulation with Julia

The sample space of a random experiment is $\{a, b, c, d, e\}$ with probabilities 0.1, 0.1, 0.2, 0.4, 0.2 respectively. Let A denote the event $\{a, b, c\}$, and let B denote the event $\{c, d, e\}$. Determine the following:

(a) $P(A)$

To calculate this we add the probabilities of getting a, b and c, $P(A) = 0.1 + 0.1 + 0.2 = 0.4$

(b) $P(B)$

To calculate this we add the probabilities of getting c, d and e, $P(B) = 0.2 + 0.4 + 0.2 = 0.8$. Note the events A and B are not disjoint as they both contain c

(c) $P(\overline{A})$

This is the probability of event A so subtract the answer from part (a) from one, $P(\overline{A}) = 1 - 0.4 = 0.6.$

(d) $P(A \cup B)$

This would be an event that contains all letters, so is the probability of getting a value in the sample space which is $P(A \cup B) = P(\Omega) = 1.$

(e) $P(A \cap B)$

As discussed in part (b) c is the only letter in both events so this would be the probability of getting c which is $P(A \cap B) = P(\text{“}c\text{”}) = 0.2.$

Below are two alternative Julia code blocks. In each block we generate samples with values “1”, “2”, “3” or “4” based on the weights, 0.1, 0.6, 0.2 and 0.1 respectively. Notice that the code blocks differ.

In [1]:
```
# code block (1)

using StatsBase
w = weights([0.1,0.6,0.2,0.1])

prop = sum([sample(w) == 1 || sample(w) == 3 for _ in 1:10^6])/10^6
```

Out[1]: 0.279807

In [2]:
```
# code block (2)

using StatsBase
w = weights([0.1,0.6,0.2,0.1])

function eventHolds(wInstance)
    wInstance == 1 || wInstance == 3
end

prop = sum([eventHolds(sample(w)) for _ in 1:10^6])/10^6
```

Out[2]: 0.300066

(f) Run code block (1) and code block (2). Explain the difference in the results. Describe the value of “prop”. Can you explain why the output of “prop” is what it is? Hint: Calculating the expected value of “prop” (by hand) for code block (2) is much more straight forward than code block (1).

In code block (2) the function checks a single randomly chosen value against the condition of equally 1 or 3, then calculating the proportion of the iterations this occurs. As these are disjoint events (a number can not be 1 and 3 at the same time) the proportion here can be calculated as

$$P(W = 1 \cup W = 3) = P(W = 1) + P(W = 3) = 0.1 + 0.2 = 0.3$$

In code block (1) the function generates a random value, checks whether it equals 1 and then generates another random value and checks whether it equals 3. This means there are *two random variables* that need to be considered here. This means the proportion of times this happens is different to code block (2) and the events are still independent but not disjoint. So the calculation is

$$\begin{aligned} P(W_1 = 1 \cup W_2 = 3) &= P(W_1 = 1) + P(W_2 - 3) - P(W_1 = 1 \cap W_2 = 3) \\ &= 0.1 + 0.2 - 0.1 \times 0.2 \\ &= 0.3 - 0.02 = 0.28 \end{aligned}$$

(g) Modify either code block (1) or code block (2) (choose the correct one), to simulate the experiment of the question (with Ω = {a, b, c, d, e}) using 10 6 replications. Based on the simulation runs, present your estimates for the probabilities in (a)–(e) and compare then to your exact answers for (a)–(e).

As the probabilities in (a) to (e) are comparing one random value, the correct code block is code block (2). Below are the simulation results for each part

In [3]:
```
#(a)
values = ['a','b','c','d','e']
w = weights([0.1,0.1,0.2,0.4,0.2])

function eventHolds(wInstance)
    wInstance == 'a' || wInstance == 'b' || wInstance == 'c'
end

prop = sum(sum([eventHolds(sample(values,w)) for _ in 1:10^6]))/10^6
```

Out[3]: 0.399972

In [4]:
```
#(b)
function eventHolds(wInstance)
    wInstance == 'c' || wInstance == 'd'|| wInstance == 'e'
end

prop = sum(sum([eventHolds(sample(values,w)) for _ in 1:10^6]))/10^6
```

Out[4]: 0.799138

In [5]:
```
#(c)
function eventHolds(wInstance)
    wInstance == 'd' || wInstance == 'e'
end

prop = sum(sum([eventHolds(sample(values,w)) for _ in 1:10^6]))/10^6
```

Out[5]: 0.60085

In [6]:
```
#(d)
function eventHolds(wInstance)
    wInstance == 'a' || wInstance == 'b' || wInstance == 'c' || wInstance == 'd'
|| wInstance == 'e'
end

prop = sum(sum([eventHolds(sample(values,w)) for _ in 1:10^6]))/10^6
```

Out[6]: 1.0

In [7]:
```
#(e)
function eventHolds(wInstance)
    wInstance == 'c'
end

prop = sum(sum([eventHolds(sample(values,w)) for _ in 1:10^6]))/10^6
```

Out[7]: 0.199844

Question 4 - NiCd Battery

In an NiCd battery, a fully charged cell is composed of Nickelic Hydroxide. Nickel is an element that has multiple oxidation states and that is usually found in the following states:

Nickel Charge	Proportions Found
0	0.12
+2	0.40
+3	0.30
+4	0.18

For both of these items, formulate the events as sets as you present your answer.

(a) What is the probability that a cell has at least two of the positive nickel-charged options?

First note that the charge of a cell is a disjoint independent event. This is the probability of having any charge other than zero,

$$P(\Omega \backslash \{0\}) = P(\{+2, +3, +4\}) = 1 - 0.12 = 0.88$$

(b) What is the probability that a cell is not composed of a positive nickel charge greater than +2?

This is the probability that a cell has +3 or +4 charge.

$$P(\{+3, +4\}) = P(+3) \cup P(+4) = P(+3) + P(+4) = 0.30 + 0.18 = 0.48$$

So taking this away from 1 we get the probability of having a cell which is not composed of a positive nickel charge greater than +2 is $1 - 0.48 = 0.52$.

Question 5 - Hacking the NSA

A computer system uses passwords that contain exactly eight characters, and each character is one of 26 lower-case letters (a-z) or 26 upper-case letters (A-Z) or 10 integers (0-9). Let Ω denote the set of all possible passwords, and let A and B denote the events that consist of passwords with only letters or only integers, respectively. Suppose that all passwords Ω are equally likely. Determine the probability of each of the following:

(a) A

First we need to know the number of possible passwords that can be generated with these characters. There are 62 possible characters (26+26+10=62) and each one is equally likely to be selected as a character of the password. This means there are $62^8 = 218340105584896$ possible passwords. Event A is the event of a password of only letters. The probability of any character in the password being a letter is $P(L) = \frac{52}{62}$. Each character can be considered an independent event so we raise this to the power of eight to get the probability. so

$$P(A) = \left(\frac{52}{62}\right)^8 = 0.2448.$$

(b) B

Using a similar method to part (a), the probability of a character being a number is $P(N) = \frac{10}{62}$. Again this is raised to the power of eight to get the probability of the event B

$$\left(\frac{10}{62}\right)^8 = 4.580 \times 10^{-7}.$$

(c) A password contains at least 1 integer.

This event is $P(\overline{A}) = 1 - P(A)$ so taking the value from part (a) we get a probability of

$$P(\overline{A}) = 0.7552$$

(d) A password contains exactly 2 integers.

This is a little more involved as there are many combinations of letters and numbers that could form a password with 2 integers. Consider the distribution of letters in the password as a binomial distribution with probability of success 0.2448. To calculated the probability we perform the following calculation,

$$\begin{aligned} P(2 \text{ integers}) &= \binom{8}{6} P(L)^6 P(N)^2 \\ &= \frac{8!}{6!2!} 0.2448^6 \times 0.7552^2 \\ &= 28 \times 0.2448^6 \times 0.7552^2 \\ &= 0.2535 \end{aligned}$$

The following Julia code generates 100 random passwords and counts how many of them contain 1 or less lower case letters.

In [8]:
```
passLength = 8
numToCheck = 1
possibleChars = ['a':'z';'A':'Z';'0':'9']

# Define a function that counts how many characters are lower case
numLowerCaseChars(str) = sum([islower(char) for char in str])

n= 100
passwords = [String(rand(possibleChars,passLength)) for _ in 1:n]
proportion = sum([numLowerCaseChars(p) <= numToCheck for p in passwords])/n
```

Out[8]: 0.09

(e) In your view, are 100 passwords sufficient for obtaining a sensible estimate for the event of having 1 or less lower case characters? Modify the code to obtain a more accurate estimate.

The probability of a character being a lower case character is $\frac{26}{62}$ and so the probability of obtaining a password with 1 or less lower case characters is

$$\begin{aligned} P(\leq 1 \text{ lower case letter}) &= P(0 \text{ lower case letters}) + P(1 \text{ lower case letter}) \\ &= \left(1 - \frac{26}{62}\right)^8 + 8 \times \frac{26}{62} \times \left(1 - \frac{26}{62}\right)^7 \\ &= 0.0876. \end{aligned}$$

Given the size of the sample space, of the order of 10^{14}, it is not sensible to obtain an estimate from only 100 passwords. A reasonable sample would be 10^6 passwords as shown below:

In [9]:
```
passLength = 8
numToCheck = 1
possibleChars = ['a':'z';'A':'Z';'0':'9']

# Define a function that counts how many chaacters are lower case
numLowerCaseChars(str) = sum([islower(char) for char in str])

n= 10^6
passwords = [String(rand(possibleChars,passLength)) for _ in 1:n]
proportion = sum([numLowerCaseChars(p) <= numToCheck for p in passwords])/n
```

Out[9]: 0.087799

(f) Modify the code to obtain estimates for the probabilities of the events in (a)-(d). Compare with your theoretical results. You may want to use the "`isnumber()`" function. e.g. `isnumber('7')`.

In [10]:
```
#(a)
numToCheck = 0

numNumbers(str) = sum([isnumber(char) for char in str])

n= 10^8
passwords = [String(rand(possibleChars,passLength)) for _ in 1:n]
proportion = sum([numNumbers(p) == numToCheck for p in passwords])/n
```

Out[10]: 0.24479904

In [11]:
```
#(b)
numToCheck = 8

proportion = sum([numNumbers(p) == numToCheck for p in passwords])/n
```

Out[11]: 3.6e-7

In [ ]:
```
#(c)
numToCheck = 1

proportion = sum([numNumbers(p) >= numToCheck for p in passwords])/n
```

In [ ]:
```
#(d)
numToCheck = 2

proportion = sum([numNumbers(p) == numToCheck for p in passwords])/n
```

When comparing these result to those calculated analytically most results look similar. The result for part (b) however is slightly different but this is due to the level of floating point numbers in Julia.

Question 6 - Cast Aluminium

Samples of a cast aluminium part are classified on the basis of surface finish (in microinches) and length measurements. The results of 200 parts are summarised as follows:

		Length	
		Excellent	Good
Surface Finish	Excellent	140	5
	Good	30	25

Let A denote the event that a sample has excellent surface finish, and let B denote the event that a sample has excellent length. Determine:

(a) $P(A)$

This is the probability that a part has excellent surface finish so the number parts with an excellent surface finished (140+5) is divided by the total number of parts (200). That is

$$P(A) = \frac{140+5}{200} = 0.725$$

(b) $P(B)$

This is the probability of excellent length. 170 parts have excellent length so the probability is

$$P(B) = \frac{170}{200} = 0.85$$

(c) $P(A|B)$

This is the probability of a part with excellent length having excellent surface finish. Using the following formula (Bayes Rule)

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{140}{170} \approx 0.8235$$

(d) $P(B|A)$

This is the probability of a part with excellent surface finish having excellent length. As above

$$P(B|A) = \frac{P(B \cap A)}{P(A)} = \frac{140}{145} \approx 0.9655$$

(e) If the selected part has excellent surface finish, what is the probability that the length is excellent?

This is the same as (d) so $P(B|A) \approx 0.9655$

(f) If the selected part has good length, what is the probability that the surface finish is excellent?

This is the probability $P(A|\overline{B})$ so following the same procedure as above

$$P(A|\overline{B}) = \frac{P(A \cap \overline{B})}{P(\overline{B})} = \frac{5}{30} = 0.1\dot{6}$$

Question 7 - Cotton Fabric and Nylon Fabric

Suppose 2% of cotton fabric rolls and 3% of nylon fabric rolls contain flaws. Of the rolls used by a manufacturing company, 20% are cotton and 80% nylon. What is the probability that a randomly selected roll used by the company contain flaws?

Let the event that these is a flaw be F, the event that a roll of cotton is selected be C and the event that a roll of nylon is select is N. The probability of a flaw can now be calculated by taking the union of the event that a flaw is present and a roll of cotton is selected and the even that a flaw is present and a roll of nylon is select. Hence:

$$\begin{aligned} P(F) &= P((F \cap C) \cup (F \cap N)) \\ &= P(F \cap C) + P(F \cap N) \quad \text{As disjoint events} \\ &= P(F|C)P(C) + P(F|N)P(N) \\ &= 0.02 \times 0.2 + 0.03 \times 0.8 \\ &= 0.028 \end{aligned}$$

Question 8 - Computer Keyboard Failure

Computer keyboard failures are due to faulty electrical connects (10%) or mechanical defects (90%). Mechanical defects are related to loose keys (22%) or improper assembly (78%), Electrical connect defects are caused by defective wires (35%), improper connections (13%), or poorly welded wires (52%).

(a) Find the probability that a failure is due to loose keys.

As 22% of Mechanical defects are caused be loose keys and 90% of defects are mechanical to work out the probability of a failure due to loose keys we have to multiply these proportions together.

$$P(LK) = P(LK|MD)P(MD) = 0.22 \times 0.9 = 0.198$$

(b) Find the probability that a failure is due to improperly connected or poorly welded wires

Here we need to add the probability of a failure from a faulty electrical connect due to improperly connections and a faulty electrical connect due to poorly welded wires.

$$\begin{aligned} P(IC \cap PW) &= P(IC|FE)P(FE) + P(PW|FE)P(FE) \\ &= (P(IC|FE) + P(PW|FE))P(FE) \\ &= (0.13 + 0.52) \times 0.1 = 0.065 \end{aligned}$$