Question 1 – Joint Probability Mass Function

Consider the function $p_{X,Y}(\cdot,\cdot)$

x	y	$p_{X,Y}(x,y)$
-1.0	1.0	1/7
0.0	2.0	2/7
1.0	3.0	1/7
1.5	3.0	2/7
3.0	4.0	1/7

Determine the following:

(a) Show that $p_{X,Y}$ is a valid probability mass function.

(b) $P(X < 2.5; Y < 3)$.

(c) $P(X < 2.5)$.

(d) $P(Y < 3)$.

(e) $P(X > 0.8, Y > 3.7)$.

(f) The marginal expected values E(X), E(Y), and marginal variances V(X), V(Y).

(g) Are X and Y independent random variables?

(h) $P(X + Y \leqslant 5)$

Question 2 – Joint Probability Density Function

Determine the value of c that makes the function

$$f(x,y) = ce^{-2x-3y}$$

a joint probability density function over the range $0 < x < \infty$ and $0 < y < x$.

Determine the following:

(a) $P(X < 1, Y < 2)$,

(b) $P(1 < X < 2)$,

(c) $P(Y > 3)$,

(d) $E[X]$,

(e) $E[Y]$,

(f) the marginal probability distribution of X,

(g) $P(X|Y = 1)$,

(h) $E[X|Y = 2]$,

Question 3 – Covariance and Correlation

Let

$$f_{x,y} = \frac{1}{2\pi} e^{-\frac{1}{2}\left[(x+1.5)^2+(y-5)^2\right]},$$

for $x, y \in \mathbb{R}$

(a) Show that f is a joint density function, that is $f_{x,y} \geqslant 0$ and

$$\int_{\mathbb{R}} \int_{\mathbb{R}} f_{x,y}\ dx\ dy = 1.$$

(b) Determine the marginal density function $f_X(x)$.

(c) Determine the conditional density function $f_x(x|y = 2)$.

(d) Determine $E(X)$.

(e) Obtain $P(X > 2, Y < 5)$.

(f) Verify your answer in (d) and (e) using R code, translating the above joint probability density function into a bivariate normal distribution, using for example the R function: `dmvnorm`/`rmvnorm` from the `mvtnorm` package.

Question 4 – More Fun With Two Random Variables

Let X and Y be independent random variables with $E(X) = 3$, $V(X) = 4$, $E(Y) = 5$, $V(Y) = 9$. Let $g(X, Y) = -X + 2Y$. Determine the following:

(a) $E(g(X, Y))$,

(b) $V(g(X, Y))$,

(c) $\rho(X, Y)$

If you know additional that $X\ U(a, b)$ and $Y\ U(c, d)$ and are independent find:

(d) (a), (b), (c) based on the information given above

(e) $P(g(X, Y) < 18)$,

Question 5 – Even More Fun With Bivariate Normal Distributions

Let X and Y be independent normally distributed with mean $\mu_X = 2$ and $\mu_Y = 3$ and standard deviations $\sigma_X = 3$ and $\sigma_Y = 5$, respectively. Determine the following:

(a) $P(3X + 6Y > 15)$,

(b) $P(3X + 6Y < 30)$,

(c) $Cov(X, Y)$

(d) Verify (a) and (b) using R code, where for each case you generate a million X's and a million Y's and simulate the linear combination $3X + 6Y$.

(e) Assume now that the random variables come from another distribution (not Normal), but keep the same means and variances. Are your answers for (a), (b), (c) likely to change?

(f) Assume now that X and Y are normally distributed but are dependent with $Cov(X, Y) = 5$. Write an explicit expression using a double integral for $P(X < 5, Y > 3)$.

Question 6 – Door Casing

There width of a casing for a door is normally distributed with a mean of 24 inches and a standard deviation of 1/8 inch. The width of a door is normally distributed with a mean of 23 and 7/8 inches and a standard deviation of 1/16 inch. Assume independence.

(a) Determine the mean and standard deviation of the difference between the width of the casing and the width of the door.

(b) What is the probability that the width of the casing minus the width of the door exceeds 1/4 inch?

(c) What is the probability that the door does not fit in the casing?

Question 7 – The Beauty of Proof

Suppose hat the joint probability distribution of the continuous random variables X and Y is constant on the rectangle $0 < x < a$ and $0 < y < b$ for a, $b \in \mathbb{R}^+$.
Show mathematically that X and Y are independent. *Hint:*

(a) Recall $\int_{\Omega_X} \int_{\Omega_Y} f(x, y)\ dy\ dx = 1$

(b) Recall X, Y *are independent if* $f_x f_y = f_{xy}$.

Question 8 – Risk Analysis

A marketing company performed a risk analysis for a manufacturer of synthetic fibres and concluded that new competitors present no risk 13% of the time, moderate risk 72% of the time, and high risk otherwise.
Eight international companies are planning to open new facilities for the manufacture of synthetic fibres with the next three years. Assume the companies are independent. Let X, Y, Z denote the number of new competitors that will pose no, moderate and high risk for the interested company, respectively.
The range of the joint probability distribution of X, Y and Z is

$$\Omega = \{(x, y, z) : x + y + z = 8, x, y, z \in [0.8]\}$$

To calculate the probability mass function for these variables, use the R code given in the file `STAT2201-A3-2019a-Q8.r`.

1. Determine $P(X = 1, Y = 3, Z = 1)$.
2. Determine $P(Y \leqslant 2)$.
3. Determine $P(Z \geqslant 2 | Y = 1, X \geqslant 4)$.
4. Determine $P(Y \leqslant 2, Z \leqslant 1 | X = 5)$.
5. Determine $E[Z | X = 5]$.