THE UNIVERSITY OF QUEENSLAND
AUSTRALIA

Analysis of Engineering and Scientific Data

Semester 1 – 2019

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Review of Chapter 2

- **Sample space:** Ω.
- **Events:** $A, B, C \ldots$ (subsets of Ω).
- **Venn Diagram**.
- **Operation Laws and DeMorgan Laws**.
  a) $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$
  b) $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$
  c) $\overline{A \cup B} = \overline{A} \cap \overline{B}$
  d) $\overline{A \cap B} = \overline{A} \cup \overline{B}$

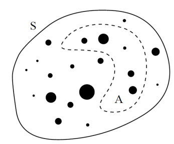
Review of Chapter 2

- **Probability:** "measure" of likelihood of an event.
- **Formal Definition**: **Definition:**
  A probability $P : \mathcal{F} \to [0, 1]$, is a rule (or function) which assigns a number between 0 and 1 to each event, and which satisfies the following axioms:
  - $0 \leq P(A) \leq 1$,
  - $P(\Omega) = 1$,
  - if $\{A_i\}$ are disjoint, then $P\left(\bigcup_i A_i\right) = \sum_i P(A_i)$.

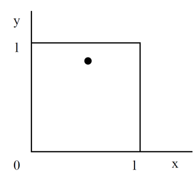
Review of Chapter 2

- A measure can be defined on both discrete and continuous state spaces.
- **geometric probability for equally likely events:**

  a) In the discrete case it will (generally) result in solving a counting problem:



  b) In the continuous case it will (generally) result in a solution of finding an area (an integration) problem:



Review of Chapter 2

- **Conditional probability:**

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)},$$

- **Chain rule:**

$$P(A \cap B) = P(A \mid B)P(B),$$

- **Law of Total Probability:** $\{B_i\}$ are a partition of Ω, then

$$P(A) = \sum_{i=1}^{n} P(A \cap B_i) = \sum_{i=1}^{n} P(A \mid B_i)\ P(B_i),$$

Review of Chapter 2

- **Independence**:

$$A, B \text{ independent iff } P(A \mid B) = P(A).$$

$$\leftrightarrow P(B \mid A) = P(B).$$

$$\leftrightarrow P(A \cap B) = P(A)\, P(B).$$

Example - independence

- Getting a 7 on the assignments (event A) if one visits the tutorials (event B)? dependent
- Being enrolled in either Calculus (event A) or Statistics (event B) in one semester. dependent
- Drawing a red ball as second ball (event A) if the first ball was blue (event B), if the balls are not returned to the bucket of coloured balls. dependent
- Getting a 2 (event A) or a 4 (event B) when rolling a dice. dependent
- Drawing a red ball as second ball (event A) if the first ball was blue (event B), if the balls are returned to the bucket of coloured balls. independent

Independence vs. Mutually Exclusive

If A, B are mutually exclusive $\longrightarrow$ A, B are independent?

NO!
For example: Rolling a dice once.
A = event to get a '2', B = event to get a '4'.
A,B are mutually exclusive but dependent since
$P(A|B) = 0$ [if you got a '4' you can not get a '2'], but $P(A) = \frac{1}{6}$, so

$$P(A|B) \neq P(A) \quad \longrightarrow A, B \text{ are dependent}$$

If A, B are mutually exclusive $\longleftarrow$ A, B are independent?

NO!
For example: Card game with 52 cards.
A = event to get a spade, B = event to get a ace.
A,B are independent since

$$\frac{1}{52} = P(A \cap B) = \frac{4}{52}\frac{1}{4} = P(B)P(A)$$

but they are not mutually exclusive because there is a card ace in spade.

The Birthday problem

What is the probability that among N students at least 2 students share the same birthday?

$P(22 \text{ students share BD}) = 1 - P(\text{no students share their BD}) = 1 - \frac{365}{365} \cdot \frac{364}{365} \cdot \frac{363}{365} \cdot \dots \cdot \frac{365-N+1}{365} = 1 - \frac{365!}{(365-N)!} \cdot \frac{1}{365^N}$

Recall: $365! = 365 \cdot 354 \cdot 363 \cdots 2 \cdot 1$

Optional: See RMarkdown document in Lecture 2 folder (blackboard and course webpage) to see this example coded example in R.

Chapter 3 & Chapter 4

1. Random Variables
2. Probability Mass/Density Function
3. Cumulative Distribution Function
4. Expectation and Variance
5. Special Discrete Distributions (Bernoulli, Binomial,Geometric)
6. Special Continuous Distributions (Uniform, Exponential, Normal)

Random Variables

Definition:
A function $X(\omega)$ assigning to every outcome ω in the sample space Ω a real number is called a **random variable**.

Example:

- Tossing a coin: $\Omega = \{H, T\}$; X could be $X(H) = 1$ (e.g. "Head is success") and $X(T) = 0$ (e.g. "Tail is failure").
- N transistors, a_i = event that i^{th} transistor works; X could assign a 1 if the transistor works, else 0, i.e., $X(a_i) = 1, \quad X(\overline{a}_i) = 0$.
- N transistors, a_i = event that i^{th} transistor works. Take 2 transistors, then the possible outcomes are $\Omega = \{(a_1, a_2), (\overline{a}_1, a_2), (a_1, \overline{a}_2), (\overline{a}_1, \overline{a}_2)\}$. X could be the number of defective transistors, i.e.,
  $X((a_1, a_2)) = 0$, $X((\overline{a}_1, a_2)) = 1 = X((a_1, \overline{a}_2))$, $X(\overline{a}_1, \overline{a}_2) = 2$.

Some notational abbreviations

- Random variables denoted by $X, Y, Z \ldots$.
- Outcomes of X are denoted by $x_1, x_2, x_3, \ldots$.
- $X = X(\omega)$.
- $\{X \leq x\} = \{\omega \in \Omega \; ; \; X(\omega) \leq x\}$.
- $\{X = x\} = \{\omega \in \Omega \; ; \; X(\omega) = x\}$.
- $P(\{\omega \in \Omega \; ; \; X(\omega) \leq x\}) = P(X \leq x)$.
- $P(\{\omega \in \Omega \; ; \; X(\omega) = x\}) = P(X = x)$.

Example:

We flip a die 15 times, such that the outcome "2" is associated with a success (=1), else failure (=0).

$\Omega = \{0,1\}^{15}$

Suppose X counts the successes:

$$X(\boldsymbol{\omega}) = \omega_1 + \omega_2 + \cdots + \omega_{15}.$$

$\{X = k\}$ is the set of outcomes with exactly k successes.

What is $P(X = k)$?

$$P(X = k) = \binom{15}{k} \left(\frac{1}{6}\right)^k \left(\frac{5}{6}\right)^{15-k},$$

Example - Generalization

We can generalize the previous example:

Let $\Omega = \{0,1\}^n$, where $\omega = 1$ represents a success, and $\omega = 0$ represents a failure.

Suppose X counts the successes of a n-times repeated experiment:

$$X(\boldsymbol{\omega}) = \omega_1 + \omega_2 + \cdots + \omega_n.$$

Let p be the probability of success in each experiment.

$\{X = k\}$ is the set of outcomes with exactly k successes.

Then

$$P(X = k) = \binom{n}{k} (p)^k (1-p)^{n-k},$$

$\longrightarrow$ **Binomial distribution**

Range of a random variable

The set of all possible values of (R.V.) X = range of $X = \Omega$

- **Discrete** random variables can only take isolated values.

  Usually counts, e.g. number of successes in a game, number of green balls in a bucket etc.

- **Continuous** random variables can take values in an interval.

  For example: rainfall measurements, lifetimes of components, lengths, etc. (at least in principle continuous).

Probability Mass Function - Discrete R.V.

Definition [Probability mass function]:
For a **discrete** random variable X, the function $P(X = x)$ is called the **probability mass function** (pmf) of X. We have for any $A \subseteq \Omega = \{x_1, \ldots, x_n\}$,

$$P(X \in A) = \sum_{x \in A} P(X = x) = \sum_{x_i \in A} P(X = x_i) = \sum_{x_i \in A} p_i,$$

with

1. $p_i = P(X = x_i) \geq 0$ for all x_i,
2. $\sum_{i=1}^{n} p_i = \sum_{i=1}^{n} P(X = x_i) = 1$.

Example: Fair die

Toss a die and let X be its face value.
X has discrete range $\Omega = \{1, 2, 3, 4, 5, 6\}$.
The probability mass function (pmf) of X is

x	1	2	3	4	5	6	Σ
$p(x) = P(X = x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	1

Example: Maximum of two fair dice
Toss two dice and let Y be the largest face value showing.
Y has a discrete range $\Omega = \{1, 2, 3, 4, 5, 6\}$.
The probability mass function (pmf) of Y is

y	1	2	3	4	5	6	Σ
$p(y) = P(Y = y)$	$\frac{1}{36}$	$\frac{3}{36}$	$\frac{5}{36}$	$\frac{7}{36}$	$\frac{9}{36}$	$\frac{11}{36}$	1

What is $P(Y > 4)$?

$P(Y > 4) = P(Y = 5) + P(Y = 6) = \frac{9+11}{36} = \frac{5}{9}$.

Probability Density Function - Continuous R.V.

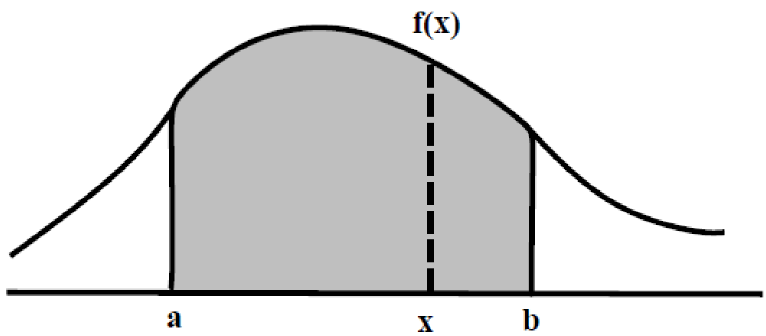
Definition [Probability density function]:
For a **continuous** random variable X, the function $f(x)$ is called the **probability density function** (pdf) of X. We have for any $A = [a_1, b_1] \subseteq \Omega = [a, b]$,

$$P(X \in A) = P(a_1 \le x \le b_1) = \int_{a_1}^{b_1} f(x)\,\mathrm{d}x$$

with

1. $f(x) \ge 0$ for all $x \in \Omega$,
2. $\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = 1$.

The function f is called the probability density function (pdf) of X.



Example:

Let $f(x) = exp(-\frac{(x-350)}{350})$.

Can f be a probability density function for the R.V. measuring the mega-litres in the Fitzroy river with $\Omega = [0, 350]$?

- $f(x) \ge 0$
-

$$\int_{x\in\Omega} f(x)\mathrm{d}x = \int_0^{350} e^{-\frac{(x-350)}{350}}\,\mathrm{d}x$$
$$= 350\left(e^{-\frac{(0-350)}{350}} - e^{-\frac{(350-350)}{350}}\right)$$
$$= 350\left(e^1 - 1\right) \approx 601 \neq 1$$

$\longrightarrow f$ is not a probability density function.

Example:

Let $f(x) = \frac{2x}{350^2}$.

Can f be a probability density function for the RV measuring the mega-litres in the Fitzroy river with $\Omega = [0, 350]$?

- $f(x) \geq 0$
- $\int_{x \in \Omega} f(x) \mathrm{d}x = \int_0^{350} \frac{2x}{350^2} \, \mathrm{d}x = \frac{2}{350^2} \int_0^{350} x \, \mathrm{d}x = \frac{2}{350^2} \left(\frac{x^2}{2} \right) |_0^{350} = 1$

$\longrightarrow$ f is a probability density function.

Summary: Probability Mass/Density Function

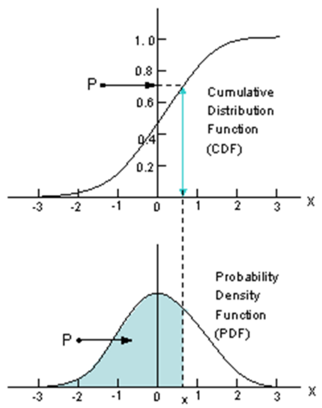
Ω

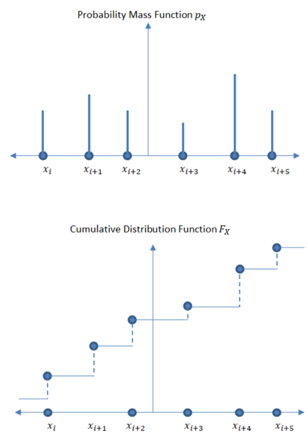
$\swarrow \qquad \searrow$

Discrete	Continuous
pmf: p_i	**pdf** f: $P(a \leq x \leq b) = \int_a^b f(x) \, \mathrm{d}x$
$p_i \geq 0$	$f(x) \geq 0$ for all $x \in \Omega$
$\sum_{i=1}^n p_i = 1$	$\int_\Omega f(x) \, \mathrm{d}x = 1$

Cumulative Distribution Function

Definition [Cumulative Distribution Function]:
The **cumulative distribution function** (cdf) of a random variable X is the function F defined by:





Source: http://reliability.srv.ualberta.ca/fundamental-probability-concepts-theory/random-variables

Source: http://home.ubalt.edu/ntsbarsh/business-stat/opre504.htm

Properties of cumulative distribution function

$$F(x) = P(X \leq x).$$

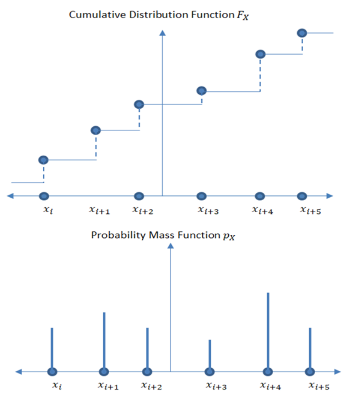
1. $0 \leq F(x) \leq 1$.
2. F is increasing: $x \leq y \quad \Rightarrow \quad F(x) \leq F(y)$.
3. It holds that $lim_{x \to \infty} F(x) = 1$, and $lim_{x \to -\infty} F(x) = 0$.
4. F is right-continuous: $lim_{h \to 0^+} F(x+h) = F(x)$.

Property for Discrete R.V.

X discrete R.V.

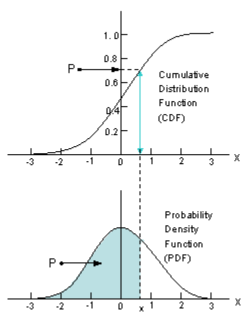
Observation:

- cdf F is a step-function.
- cdf F jumps at all points $x \in \Omega$.
- The height of the jump at x_i is p_i.



Property for Continuous R.V.

X continuous R.V.



Source: http://home.ubalt.edu/ntsbarsh/business-stat/opre504.htm

Observation:

- cdf F is continuous.
- By definition of F
  $F(x) = P(X \le x) = \int_{-\infty}^{x} f(x)\,\mathrm{d}x$, so $f(x) = \frac{\mathrm{d}}{\mathrm{d}x}F(x)$.

Example:

Draw a random number from the interval of real numbers [1, 3]. Let X represent the number.
Each number is equally possible.
$\longrightarrow$ uniform distribution.

What is the cdf F of X?

For example $F(2.8) = P(X \leq 2.8) = P(1 \leq X \leq 2.8) \overset{geometric\, prob.}{=} \frac{2.8-1}{3-1} = \frac{1.8}{2}$

to be continued next lecture.