THE UNIVERSITY OF QUEENSLAND

AUSTRALIA

Analysis of Engineering and Scientific Data

Semester 1 – 2019

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Example:

Draw a random number from the interval of real numbers [1, 3].
Let X represent the number.
Each number is equally possible. $\longrightarrow$ uniform distribution.
What is the cdf F of X?

Recall, the pdf of the uniform distribution is

$$f(x) = \begin{cases} \frac{1}{2} & 1 \leq x \leq 3 \\ 0 & \text{else.} \end{cases}$$

Therefore

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(x)\,\mathrm{d}x = \begin{cases} 1 & x > 3 \\ \frac{1}{2}(x-1) & 1 \leq x \leq 3 \\ 0 & x < 1. \end{cases}$$

Is any of these F a cdf of a continuous R.V.?

$$a) \quad F(x) = \begin{cases} 0 & x < -1 \\ 0.3 & -1 \leq x < 1 \\ 1 & x \geq 1, \end{cases} \qquad b) \quad F(x) = \begin{cases} 0 & x < 0 \\ \frac{x}{2} & 0 \leq xleq1 \\ 1 & x > 1. \end{cases}$$

Check the properties:

1. $0 \leq F(x) \leq 1$,
2. F is increasing,
3. $\lim_{x\to\infty} F(x) = 1$ and $\lim_{x\to-\infty} F(x) = 0$,
4. $\lim_{h\to 0^+} F(x+h) = F(x)$.

$$a) \quad F(x) = \begin{cases} 0 & x < -1 \\ 0.3 & -1 \leq x < 1 \\ 1 & x \geq 1, \end{cases}$$

1. $0 \leq F(x) \leq 1,$ ✓
2. F is increasing, ✓
3. $\lim_{x\to\infty} F(x) = 1$ ✓ and $\lim_{x\to-\infty} F(x) = 0$ ✓,
4. $\lim_{h\to 0^+} F(x+h) = F(x)$

   at $x = -1$: $$\underbrace{\lim_{h\to 0} F(-1+h)}_{0.3} \overset{✓}{=} \underbrace{F(-1)}_{0.3}$$

   at $x = 1$: $$\underbrace{\lim_{h\to 0} F(1+h)}_{1} \overset{✓}{=} \underbrace{F(1)}_{1}$$

   All conditions are satisfied, so in fact F is a cdf.

$$b) \quad F(x) = \begin{cases} 0 & x < 0 \\ \frac{x}{2} & 0 \le x \le 1 \\ 1 & x > 1. \end{cases}$$

1. $0 \le F(x) \le 1$, ✓
2. F is increasing, ✓
3. $\lim_{x\to\infty} F(x) = 1$ ✓ and $\lim_{x\to-\infty} F(x) = 0$ ✓,
4. $\lim_{h\to 0^+} F(x+h) = F(x)$

at $x = 0$: $\underbrace{\lim_{h\to 0} F(h)}_{\frac{h}{2}\to 0} \overset{✓}{=} \underbrace{F(0)}_{\frac{0}{2}=0}$

at $x = 1$: $\underbrace{\lim_{h\to 0} F(1+h)}_{1} \overset{\times}{=} \underbrace{F(1)}_{\frac{1}{2}}$

The last condition is not satisfied, so F is not a cdf.

The cdf and/or pmf/pdf specify the distribution of a R.V. X.

Crucial values for a distribution of X:

- $\mathbb{E}[X]$ = **expectation of a random variable**

  $\sim$ "weighted average" of values of X, weighted by their probability

- $\mathrm{Var}(X)$ = **variance of a random variable**

  $\sim$ "spread/dispersion" around $\mathbb{E}[X]$ of the distribution.

Expectation of a R.V.

Definition (Expectation)

Let X be a R.V. with pmf P/pdf f.
$\mathbb{E}[X]$= the expectation (or expected value/mean value) of X, calculated as:

- X is discrete R.V.:

$$\mathbb{E}[X] = \mu_X = \sum_{x_i \in \Omega} x_i \, P(X = x_i).$$

- X is continuous R.V.:

$$\mathbb{E}[X] = \mu_X = \int_{x \in \Omega} x f(x) \, \mathrm{d}x.$$

Expectation - Example

- What is the expected value of a tossing of a fair die?

Note, $x_1 = 1, x_2 = 2, \ldots, x_6 = 6$ and

$$P(X = 1) = \cdots = P(X = 6) = \frac{1}{6},$$

we have

$$\mathbb{E}[X] = \sum_{x_i \in \Omega} x_i \; P(X = x_i) = \sum_{i=1}^{6} i \cdot \frac{1}{6} = 7/2.$$

Note: $\mathbb{E}[X]$ is not necessarily a possible outcome of X.

Expectation - Example

- What is the expected value of a uniform distributed R.V. X on $(0, 50]$?

  Recall pdf of uniform distributed R.V. X is:

  $$f(x) = \begin{cases} 0 & x \in (-\infty, 0] \cup (50, \infty) \\ \frac{1}{50} & x \in (0, 50] \end{cases}$$

  Therefore,

  $$\begin{aligned} \mathbb{E}[X] &= \int_{-\infty}^{\infty} x f(x) \, \mathrm{d}x = \int_{0}^{50} x \frac{1}{50} \, \mathrm{d}x \\ &= \frac{1}{50} \cdot \frac{1}{2} \left(50^2 - 0\right) = 25 \end{aligned}$$

Expectation - Interpretation I

Interpretation of $\mathbb{E}[X]$ as "expected profit":

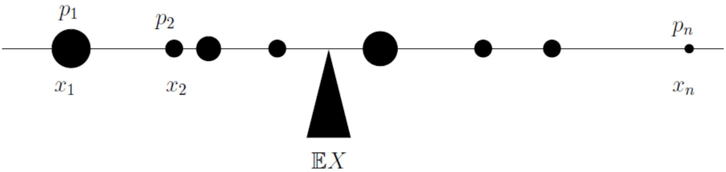
Suppose we play a game where you throw two dice.
You are getting paid the sum of the dice in \$.
To enter the game you must pay z\$.
What would be a "fair" amount for z?

$$
\begin{aligned}
z =& \mathbb{E}[X] = \sum_{i=2}^{12} i \cdot P(X = i) = \\
=& 2 \cdot P(X = 2) + \ldots + 12 \cdot P(X = 12) = \\
=& 2 \cdot \frac{1}{36} + 3 \cdot \frac{3}{36} + \ldots + 12 \cdot \frac{1}{36} = 7.
\end{aligned}
$$

Expectation - Interpretation II

Interpretation of $\mathbb{E}[X]$ as "Centre of Mass":

Imagine point masses with weights $p_1, p_2, \ldots, p_n$ are placed at positions $x_1, x_2, \ldots, x_n$:



Centre of mass = place where we can "balance" the weights

$$= x_1 p_1 + \cdots + x_n p_n = \sum_{x_i} x_i p_i = \sum_{x_i} x_i P(X = x_i) = \mathbb{E}[X].$$

Expectation of Transformations

Definition (Expectation of a function of X)

For any $g : \mathbb{R} \to \mathbb{R}$,

- if X is a discrete R.V., then

$$\mathbb{E}[g(X)] = \sum_{x_i \in \Omega} g(x_i)\ P(X = x_i),$$

  where P is the pmf of X.

- if X is a continuous R.V., then

$$\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(x) f(x) \mathrm{d}x,$$

  where f is the pdf of X.

Example:

Let X be the outcome of the toss of a fair die. Find $\mathbb{E}\left[X^2\right]$.

$$\mathbb{E}\left[X^2\right] = \sum_{i=1}^{6} x_i^2 P(X = x_i) = 1^2 \cdot \frac{1}{6} + 2^2 \cdot \frac{1}{6} + \ldots + 6^2 \cdot \frac{1}{6} = \frac{91}{6}.$$

Note:

$$\frac{91}{6} = \mathbb{E}\left[X^2\right] \neq (\mathbb{E}[X])^2 = \left(\frac{7}{2}\right)^2 = \frac{49}{4}.$$

Example:

Let X be uniformly distributed in $(0, 2\pi]$, and let $Y = g(X) = a\ cos(\omega t + X)$ be the value of a sinusoid signal at time t (i.e., X is like a uniform random phase).

Find the expected value $\mathbb{E}[Y]$.

$$\mathbb{E}[Y] = \mathbb{E}[a\ cos(\omega t + X)] = \int_0^{2\pi} acos(\omega t + x)\frac{1}{2\pi}\mathrm{d}x$$
$$= a\frac{1}{2\pi}sin(\omega t + x)\ |_0^{2\pi} = 0.$$

Properties of $\mathbb{E}$

1. $\mathbb{E}[aX + b] = a\mathbb{E}[X] + b$.
2. $\mathbb{E}[g(X) + h(X)] = \mathbb{E}[g(X)] + \mathbb{E}[h(X)]$.

Proof.

1. Wlog X continuous with pdf $f(x)$ (discrete case similar)

$$\begin{aligned}\mathbb{E}[aX + b] &= \int_\Omega (ax + b) f(x) \mathrm{d}x = a \int_\Omega x f(x) \mathrm{d}x + b \int_\Omega f(x) \mathrm{d}x \\ &= a\mathbb{E}[X] + b \cdot 1 = a\mathbb{E}[X] + b.\end{aligned}$$

2. $\mathbb{E}[g(X) + h(X)] = \int (g(x) + h(x)) f(x) \mathrm{d}x = \int g(x) f(x) \mathrm{d}x + \int h(x) f(x) \mathrm{d}x = \mathbb{E}[g(X)] + \mathbb{E}[h(X)]$.

$\square$

Variance of a R.V.

Definition (Variance)

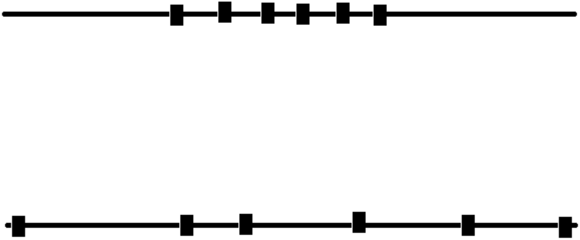
The variance of a random variable X, denoted by $\mathrm{Var}(X)$ is defined by

$$\mathrm{Var}(X) = \sigma_X^2 = \mathbb{E}[X - \mathbb{E}[X]]^2.$$

It may be regarded as a measure of the consistency of outcome: a smaller value of $\mathrm{Var}(X)$ implies that X is more often near $\mathbb{E}[X]$.

The square root of the variance is called the <u>standard deviation</u>.

The variance of a random variable



Properties of variance

1. $\mathrm{Var}(X) = \mathbb{E}\left[X^2\right] - (\mathbb{E}\left[X\right])^2$.
2. $\mathrm{Var}(aX + b) = a^2\mathrm{Var}(X)$.

Proof.

1. $$\begin{aligned}\mathrm{Var}(X) &= \mathbb{E}\left[(X - \mu_X)^2\right] = \mathbb{E}\left[X^2 - 2X\mu_X + \mu_X^2\right] \\ &= \mathbb{E}\left[X^2\right] - 2\mu_X(\mathbb{E}[X]) + \mu_X^2 = \mathbb{E}\left[X^2\right] - \mu_X^2.\end{aligned}$$

2. $$\begin{aligned}\mathrm{Var}(aX + b) &= \mathbb{E}\left[(aX + b - (a\mu_X + b))^2\right] \\ &= \mathbb{E}\left[a^2(X - \mu_X)^2\right] = a^2\mathrm{Var}(X).\end{aligned}$$

□

Moments of a random variable

- $\mathbb{E}[X] = \mathbb{E}\left[X^1\right]$
- $\mathbb{E}\left[X^2\right] = \mathrm{Var}(X) + \left(\mathbb{E}\left[X^1\right]\right)^2$
- $\mathbb{E}\left[X^r\right] =$ "r-th moment of X":
  - If X is discrete: $\mathbb{E}[X^r] = \sum_{i=1}^{n} x_i^r p_i$
  - If X is continuous: $\mathbb{E}[X^r] = \int_\Omega x^r f(x)\,\mathrm{d}x$

Remark: However, note that the expectation, or any moment of a random variable, need not always exist (could be $\pm\infty$).

Important Distributions:

- for a discrete R.V. X (discrete distribution)
  1. Bernoulli Distribution
  2. Binomial Distribution
  3. Geometric Distribution
- for a continuous R.V. X (continuous distribution)
  1. Uniform Distribution
  2. Exponential Distribution
  3. Normal Distribution

Discrete Distribution I

Bernoulli Distribution

Discrete Distribution I - Bernoulli

R.V. X has a **Bernoulli distribution** with success probability p, denoted by $X \sim \text{Ber}(p)$, if $X \in \{0, 1\}$ and

$$P(X = 1) = p_1 = p \qquad P(X = 0) = 1 - P(X = 1) = (1 - p)$$

It models for example:

- a single coin toss experiment,
- a success or a failure of message passing,
- a success of a certain drug,
- or, randomly selecting a person from a large population, and ask if she votes for a certain political party.

Bernoulli Distribution - Properties

- The expected value is

$$\mathbb{E}[X] = 1 \times p + 0 \times (1 - p) = p.$$

- The variance is

$$\mathrm{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = 1^2 \times p + 0^2 \times (1 - p) - p^2 = p(1 - p).$$

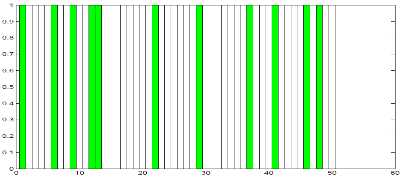
Discrete Distribution II

Binomial Distribution

Discrete Distribution II - Binomial

Binomial distribution = Sequence of independent Bernoulli trials, denoted by $X \sim \text{Bin}(n, p)$.

- $\Omega = \{0, 1\}^n$
- X = number of success in n trials,
- p = probability of success (for each Bernoulli trial).



Pmf of X:

$$P(X = x) = \binom{n}{x} p^x (1-p)^{n-x}, \qquad x = 0, 1, 2, \ldots, n$$

Binomial Distribution - Properties

- $\mathbb{E}[X] = \sum_{i=0}^{n} x_i p_i = \sum_{x=0}^{n} x \binom{n}{x} p^x (1-p)^{n-x}$
  Alternatively:

  $$X = X_1 + X_2 + \cdots + X_n,$$

  where $X_i \sim \text{Ber}(p)$, so

  $$\mathbb{E}[X] = \mathbb{E}[X_1 + X_2 + \cdots + X_n] = \sum_{i=1}^{n} \mathbb{E}[X_i] = np.$$

- $\mathrm{Var}(X) = \mathrm{Var}(X_1 + X_2 + \cdots + X_n) = \sum_{i=1}^{n} \mathrm{Var}(X_i) = np(1-p).$

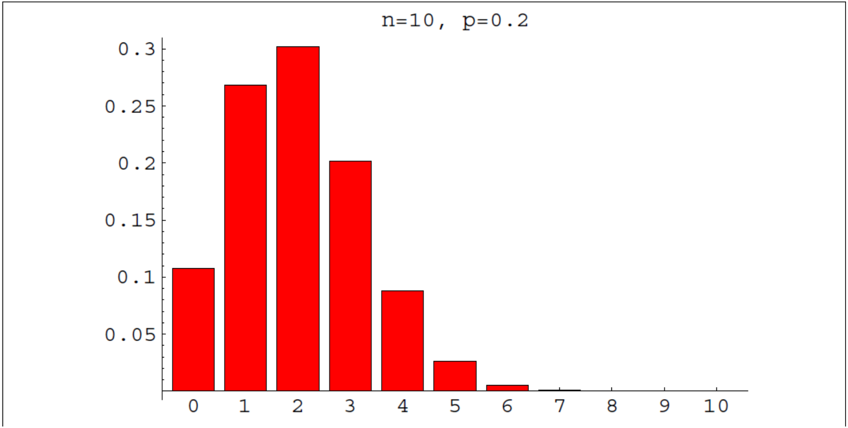
Example (Binomial Distribution):

In a country, 51% favours party A and 49% favours party B.
What is the probability that out of 200 randomly selected individuals more people vote for B than for A?

- Let a vote for A be a "success".
- Selecting the 200 people is equivalent to performing a sequence of 200 independent Bernoulli trials with success probability 0.51.
- We are looking for the probability that we have less than 100 successes, which is

$$\sum_{i=0}^{99} \binom{200}{i} 0.51^{i} 0.49^{200-i} \approx 0.36.$$

Binomial Distribution



Discrete Distribution III

Geometric Distribution

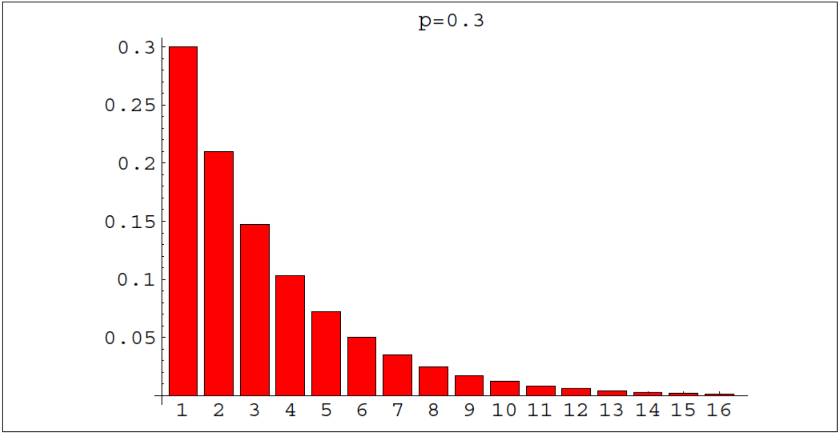
Discrete Distribution III - Geometric

Geometric Distribution:

- $\approx X$ is the number of (independent) Bernoulli trials until the first success,
- denoted by $X \sim G(p)$
- The pmf is:

$$P(X = x) = (1-p)^{x-1}p, \quad x = 1, 2, 3, \ldots$$

- $\mathbb{E}[X] = \frac{1}{p}$
- $\mathrm{Var}(X) = \frac{1-p}{p^2}$


p=0.3
0.3
0.25
0.2
0.15
0.1
0.05
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16

Example (Geometric Distribution):

Suppose you are looking for a job but the market is not very good, so the probability that the person is accepted (after a job interview) is only $p = 0.05$.
Let X be the number of interviews in which the person is rejected until he eventually finds a job.
TWhat is the probability that you need $k > 50$ interviews?

$$P(X = k) = (1-p)^{50}(1-p)^{(k-51)}p < (1-p)^{50} \sim 0.07.$$

What is the probability that you need at least 15 but less than 30 interviews?

$$P(15 \leq X < 30) = \sum_{i=15}^{29} 0.050.095^{i-1} = 0.05 \sum_{i=14}^{28} 0.95^{i} =$$

$$= 0.05 \left[\frac{1-0.95^{29}}{1-0.95} - \frac{1-0.95^{14}}{1-0.95} \right] = 0.05\,[15.4813 - 10.2465] \sim 0.2617$$