THE UNIVERSITY OF QUEENSLAND
AUSTRALIA

Analysis of Engineering and Scientific Data

Semester 1 – 2019

Sabrina Streipert s.streipert@uq.edu.au

Descriptive Statistics

- Visualisation of the data.

- Analysis and presentation of characteristics of the data.

Data types

Possible data types:

1. *Continuous quantitative* data $\longrightarrow$ values in continuous range (height, width, length, temperature, humidity, volume, area, and price)

Data types

Possible data types:

1. *Continuous quantitative* data $\longrightarrow$ values in continuous range (height, width, length, temperature, humidity, volume, area, and price)
2. *Discrete qualitative* data (*factor / categorical variable*) $\longrightarrow$ values in discrete range (number of family members, gender (male or female), count of objects).

Data types

Possible data types:

1. *Continuous quantitative* data $\longrightarrow$ values in continuous range (height, width, length, temperature, humidity, volume, area, and price)
2. *Discrete qualitative* data (*factor* / *categorical variable*) $\longrightarrow$ values in discrete range (number of family members, gender (male or female), count of objects).

   Discrete Sub-types:

Data types

Possible data types:

1. *Continuous quantitative* data $\longrightarrow$ values in continuous range (height, width, length, temperature, humidity, volume, area, and price)
2. *Discrete qualitative* data (*factor* / *categorical variable*) $\longrightarrow$ values in discrete range (number of family members, gender (male or female), count of objects).

   Discrete Sub-types:

   - *Nominal* factors = variables without order, such as males and females.

Data types

Possible data types:

1. *Continuous quantitative* data $\longrightarrow$ values in continuous range (height, width, length, temperature, humidity, volume, area, and price)
2. *Discrete qualitative* data (*factor* / *categorical variable*) $\longrightarrow$ values in discrete range (number of family members, gender (male or female), count of objects).

   Discrete Sub-types:

   - *Nominal* factors = variables without order, such as males and females.
   - *Ordinal* factors = variable with a certain order, such as *age group*.

Data configurations

- Each configuration will consist of continuous and discrete (ordinal and nominal) variables.

Data configurations

- Each configuration will consist of continuous and discrete (ordinal and nominal) variables.
- **Major configuration types:**

Data configurations

- Each configuration will consist of continuous and discrete (ordinal and nominal) variables.
- **Major configuration types:**
  - A single sample configuration consists of m scalars: $\mathcal{D} = \{x_1, x_2, \ldots, x_m\}$.
    Nr of fisherman per day; $m = 365$ and $x_i = 0, 1, \ldots$.

Data configurations

- Each configuration will consist of continuous and discrete (ordinal and nominal) variables.
- **Major configuration types:**
  - A single sample configuration consists of m scalars: $\mathcal{D} = \{x_1, x_2, \ldots, x_m\}$.
    Nr of fisherman per day; $m = 365$ and $x_i = 0, 1, \ldots$.
  - Two (or more) sets of samples:
    $$\mathcal{D} = \left\{ \left\{x_1^1, \ldots, x_{m_1}^1\right\}, \left\{x_1^2, \ldots, x_{m_2}^2\right\}, \ldots, \left\{x_1^k, \ldots, x_{m_k}^k\right\} \right\}.$$
    Nr of fisherman per day in k different regions.

Data configurations

- Each configuration will consist of continuous and discrete (ordinal and nominal) variables.
- **Major configuration types:**
  - A single sample configuration consists of m scalars: $\mathcal{D} = \{x_1, x_2, \ldots, x_m\}$.
    Nr of fisherman per day; $m = 365$ and $x_i = 0, 1, \ldots$.
  - Two (or more) sets of samples:
    $$\mathcal{D} = \left\{ \left\{x_1^1, \ldots, x_{m_1}^1\right\}, \left\{x_1^2, \ldots, x_{m_2}^2\right\}, \ldots, \left\{x_1^k, \ldots, x_{m_k}^k\right\} \right\}.$$
    Nr of fisherman per day in k different regions.
  - Data tuples: $\mathcal{D} = \{(x_{1,1}, x_{1,2}), (x_{2,1}, x_{2,2}), \ldots, (x_{m,1}, x_{m,2})\}$.
    $x_{i,1} =$ Nr of fisherman at ith day, $x_{i,2}$ is the number of fishing nets used at day i.

Data configurations

- Each configuration will consist of continuous and discrete (ordinal and nominal) variables.
- **Major configuration types:**
  - A single sample configuration consists of m scalars: $\mathcal{D} = \{x_1, x_2, \ldots, x_m\}$.
    Nr of fisherman per day; $m = 365$ and $x_i = 0, 1, \ldots$.
  - Two (or more) sets of samples:
    $$\mathcal{D} = \left\{ \left\{x_1^1, \ldots, x_{m_1}^1\right\}, \left\{x_1^2, \ldots, x_{m_2}^2\right\}, \ldots, \left\{x_1^k, \ldots, x_{m_k}^k\right\} \right\}.$$
    Nr of fisherman per day in k different regions.
  - Data tuples: $\mathcal{D} = \{(x_{1,1}, x_{1,2}), (x_{2,1}, x_{2,2}), \ldots, (x_{m,1}, x_{m,2})\}$.
    $x_{i,1}$ = Nr of fisherman at ith day, $x_{i,2}$ is the number of fishing nets used at day i.
  - Generalization of tuples to vectors:
    $$\mathcal{D} = \{(x_{1,1}, \ldots, x_{1,n}), \ldots, (x_{m,1}, \ldots, x_{m,n})\}$$
    $x_{i,1}$ = Nr of fisherman at ith day, $x_{i,2}$ is the number of fishing nets used at day i, $x_{i,3}$ = Sea-Surface temperature at day $i, \ldots$

1. Data tables

The table **rows** represent observed measurements for *independent* variables (**columns**).

Observ.	variable 1	variable 2	$\cdots$	variable i	$\cdots$	variable n
1	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
2	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
m	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$

```r
library(carData)
D <- Arrests
tail(D)

#or alterantive:
library(data.table)
print(data.table(D))
```

	released	colour	year	age	sex	employed	citizen	checks
5221	Yes	White	2002	22	Male	Yes	Yes	0
5222	Yes	White	2000	17	Male	Yes	Yes	0
5223	Yes	White	2000	21	Female	Yes	Yes	0
5224	Yes	Black	1999	21	Female	Yes	Yes	1
5225	No	Black	1998	24	Male	Yes	Yes	4
5226	Yes	White	1999	16	Male	Yes	Yes	3

Figure: Data on police arrests in Toronto for possession of marijuana.

Data summarization

A *statistic* is a numerical quantity, such as the proportion, that is computed from a sample $x_1, \ldots, x_m$.

```
library(dplyr)
D1 <- D %>% group_by(sex) %>% summarize(Count_Arrests = n(), Proportion = Count_Arrests/nrow(D))
D1
```

	sex	Count_Arrests	Proportion
	<fct>	<int>	<dbl>
1	Female	443	0.0848
2	Male	4783	0.915

Data summarization

Study a **correlation** between the two factor variables using the so called **contingency table**:

```r
D2 <- D %>% mutate(sex = ifelse(sex=="Female",1,0), employed = ifelse(employed == "Yes",1,0)) %>%
  select(sex, employed,age, year)

round(cor(D2), digits = 3)
```

	sex	employed	age	year
sex	1.000	-0.039	-0.011	-0.020
employed	-0.039	1.000	-0.117	0.030
age	-0.011	-0.117	1.000	-0.005
year	-0.020	0.030	-0.005	1.000

Summary Statistics = tool for an exploration of a variable.
Given a data vector of numbers $\mathbf{x} = (x_1, \ldots, x_n)$, we have:

Summary Statistics = tool for an exploration of a variable.
Given a data vector of numbers $\mathbf{x} = (x_1, \ldots, x_n)$, we have:

- **Sample mean**:

Summary Statistics = tool for an exploration of a variable. Given a data vector of numbers $\mathbf{x} = (x_1, \ldots, x_n)$, we have:

- **Sample mean**:

$$\bar{\mathbf{x}} = \frac{1}{n} \sum_{i=1}^{n} x_i.$$

Summary Statistics = tool for an exploration of a variable.
Given a data vector of numbers $\mathbf{x} = (x_1, \ldots, x_n)$, we have:

- **Sample mean**:

$$\bar{\mathbf{x}} = \frac{1}{n} \sum_{i=1}^{n} x_i.$$

```
D <- Arrests
mean(D$age)
> 23.84654

#or alternative:
sum(D$age)/nrow(D)
> 23.84654
```

Describing quantitative data

Describing quantitative data

- **Range** of data:

$$\mathrm{range} = \max_{1 \le i \le n} x_i - \min_{1 \le i \le n} x_i.$$

Describing quantitative data

- **Range** of data:

$$\mathrm{range} = \max_{1\le i\le n} x_i - \min_{1\le i\le n} x_i.$$

- The *order statistics*.
  First, sort the data to obtain $x_{(1)} \le x_{(2)} \le \ldots \le x_{(n)}$, and observe the following.

Describing quantitative data

- **Range** of data:

$$\text{range} = \max_{1 \leq i \leq n} x_i - \min_{1 \leq i \leq n} x_i.$$

- The *order statistics*.
  First, sort the data to obtain $x_{(1)} \leq x_{(2)} \leq \ldots \leq x_{(n)}$, and observe the following.

  1. The minimum: $x_{(1)}$.

Describing quantitative data

- **Range** of data:

$$\text{range} = \max_{1 \le i \le n} x_i - \min_{1 \le i \le n} x_i.$$

- The *order statistics*.
  First, sort the data to obtain $x_{(1)} \le x_{(2)} \le \ldots \le x_{(n)}$, and observe the following.
  1. The minimum: $x_{(1)}$.
  2. The maximum: $x_{(n)}$.

Describing quantitative data

- **Range** of data:

$$\text{range} = \max_{1\leq i\leq n} x_i - \min_{1\leq i\leq n} x_i.$$

- The *order statistics*.
  First, sort the data to obtain $x_{(1)} \leq x_{(2)} \leq \ldots \leq x_{(n)}$, and observe the following.
  1. The minimum: $x_{(1)}$.
  2. The maximum: $x_{(n)}$.
  3. The median $\tilde{\mathbf{x}} =$ "middle" of data.
     (order the data: $x_1 \leq x_2 \leq \cdots \leq x_n$):

$$\begin{cases} x_{\left(\frac{n+1}{2}\right)} & \text{if } n \text{ is odd,} \\ \frac{1}{2}\left(x_{\left(\frac{n}{2}\right)} + x_{\left(\frac{n+1}{2}\right)}\right) & \text{if } n \text{ is even.} \end{cases}$$

```
D <- Arrests

R <- max(D$age) - min(D$age)
> 54

Min_age <- min(D$age)
> 12

Max_age <- max(D$age)
> 66

Med_age <- median(D$age)
> 21
```

- **Sample Variance** (data - spread):

- **Sample Variance** (data - spread):

$$s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{\mathbf{x}})^2,$$

where $\bar{\mathbf{x}}$ is the sample mean.

- **Sample Variance** (data - spread):

$$s^2 = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{\mathbf{x}})^2,$$

where $\bar{\mathbf{x}}$ is the sample mean.

Sample Standard Deviation $= s = \sqrt{s^2}$

- **Sample Variance** (data - spread):

$$s^2 = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{\mathbf{x}})^2,$$

where $\bar{\mathbf{x}}$ is the sample mean.

Sample Standard Deviation $= s = \sqrt{s^2}$

- **Sample Correlation Coefficient:**

$$r_{\mathbf{xy}} = \frac{\sum_{i=1}^{n} (x_i - \bar{\mathbf{x}}) (y_i - \bar{\mathbf{y}})}{\sqrt{\sum_{i=1}^{n} (x_i - \bar{\mathbf{x}})^2 \ \sum_{i=1}^{n} (y_i - \bar{\mathbf{y}})^2}}$$

```
D <- Arrests
# Sample Variance
Sample_Var <-  var(D$age)
> 69.15807

#or alterantively
mean_age <- mean(D$age)
D3 <- D %>% mutate(Diff = age-mean_age, Diff_squ
    = Diff*Diff)
Sample_Var <- sum(D3$Diff_squ)/(nrow(D3)-1)
> 69.15807


# Sample Standard Deviation
sd(Sampel_Var)
> 8.316133

# or alternatively:
Sample_STD <- sqrt(Sample_Var)
> 8.316133
```

```
D <- Arrests
D4 <- D %>% mutate(sex = ifelse(sex=="Female"
    ,1,0))

# Sample Correlation Coefficient
Sample_cor <- cor(D4$age, D4$sex)
> -0.01148502

#or alterantively
mean_age <- mean(D4$age)
mean_sex <- mean(D4$sex)
D5 <- D4 %>% mutate( Num = (age-mean_age)*(sex-
    mean_sex), Denom1 = (age-mean_age)**2, Denom2
     = (sex-mean_sex)**2)

Sample_cor <- sum(D5$Num)/sqrt(sum(D5$Denom1)*
    sum(D5$Denom2))
> -0.01148502
```

Describing quantitative data

- **p-quantile** $(0 < p < 1)$
  $= z$ such that $F(z) = P(X \leq z) = p$

  Common values: 0.25, 0.5, 0.75 quantiles (=25, 50, and 75 percentiles /first, second, and third quartiles)

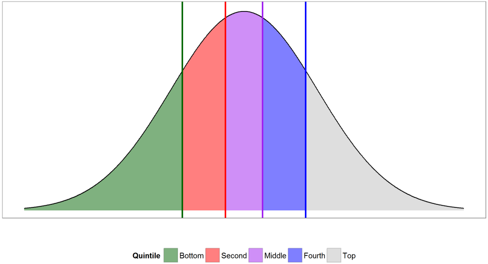
```
D <- Arrests

quantile(D$age)
```

>	0%	25%	50%	75%	100%
	12	18	21	27	66

```
quantile(D$age, seq(0,1,by=.2))   #quintile
```

>	0%	20%	40%	60%	80%	100%
	12	17	20	23	30	66



Source: https://stackoverflow.com/questions/26266246/ggplot2-stat-function-can-we-use-the-generated-y-values-on-other-layers/26280013

The quantile of a probability distribution

Let f be a prob. density function for a R.V. X.

- Given $\alpha \in [0, 1]$, what is x such that $P(X \leq x) = \alpha$?

The quantile of a probability distribution

Let f be a prob. density function for a R.V. X.

- Given $\alpha \in [0,1]$, what is x such that $P(X \leq x) = \alpha$?

- By definition:

$$P(X \leq x) = F(x) = \int_{-\infty}^{x} f(u)\mathrm{du} = \alpha.$$

Example: $X \sim Exp(1)$ and $\alpha = 0.3$, find x.

$$0.3 = \int_0^x \lambda e^{-\lambda \hat{x}} d\hat{x} = \int_0^x e^{-\hat{x}} d\hat{x} = -e^{-\hat{x}}\Big|_0^x = 1 - e^{-x}$$

$$\Rightarrow 0.3 = 1 - e^{-x} \Rightarrow x = -\log(0.7)$$

Data Analysis

- 1st step: Data-Table (+ (statistic) summary of data)
- 2nd step: **Visualisation** with the aim of:

Data Analysis

- 1st step: Data-Table (+ (statistic) summary of data)
- 2nd step: **Visualisation** with the aim of:

  1. Identifying the most common values (for each variable)

Data Analysis

- 1st step: Data-Table (+ (statistic) summary of data)
- 2nd step: **Visualisation** with the aim of:

1. Identifying the most common values (for each variable)
2. Determining the amount of variability (for each variable)

Data Analysis

- 1st step: Data-Table (+ (statistic) summary of data)
- 2nd step: **Visualisation** with the aim of:

  1. Identifying the most common values (for each variable)
  2. Determining the amount of variability (for each variable)
  3. Recognising unusual observations.

Data Analysis

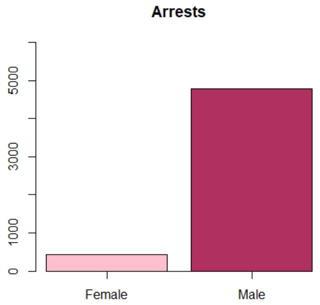
- 1st step: Data-Table (+ (statistic) summary of data)
- 2nd step: **Visualisation** with the aim of:

  1. Identifying the most common values (for each variable)
  2. Determining the amount of variability (for each variable)
  3. Recognising unusual observations.
  4. Exploring trends in the data.

Visualization of Discrete Data: Bar chart

Visualization for **factor variables**
(Nominal factor):

```
barplot(table(D$sex), main='Arrests', ylim=c
    (0,6000), axis.lty=1, col=c("Pink", "Maroon")
    )
```



Visualization of Discrete Data: Bar chart

Barplot for Ordinal factor:

```
barplot(table(D$year), main='Arrests', axis.lty
    =1, col= "Maroon")
```

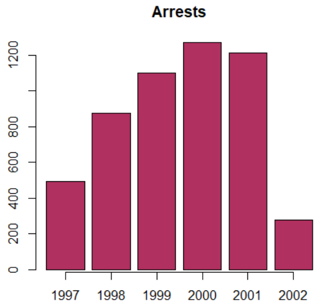
What will this code produce?

Visualization of Discrete Data: Bar chart

Barplot for Ordinal factor:

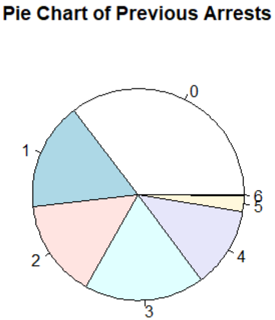
```
barplot(table(D$year), main='Arrests', axis.lty
    =1, col= "Maroon")
```

What will this code produce?



Visualization of Discrete Data: Pie chart

```
slices <- table(D$checks)
pie(slices, labels = rownames(slices), main = "
    Pie Chart of Previous Arrests")
```

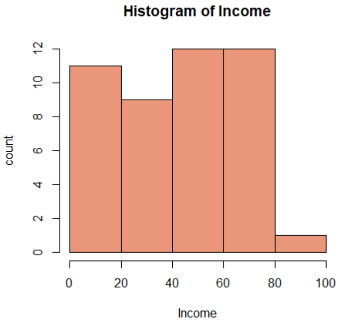


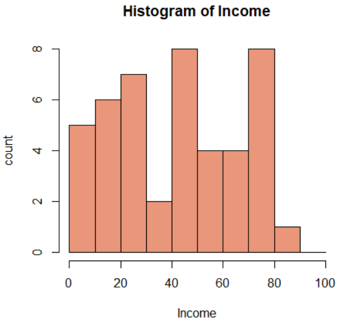
Visualization of "Continuous" Data: Histogram

Continuous analogue of bar plot

Idea:

- Divide the range of a continuous variable into interval-bins
- Plot the associated frequencies for each bin.





```r
D_c <- Duncan # data set in carData - library

#left image:
hist(D_c$income, breaks = seq(0,100,20), col="DarkSalmon", main = "Histogram of Income",
    xlab = "Income", ylab = "count")
```

change 20 to 10

How do you have to change the R-code to get the image on the right?

white	aliceblue	antiquewhite	antiquewhite1	antiquewhite2
antiquewhite3	antiquewhite4	aquamarine	aquamarine1	aquamarine2
aquamarine3	aquamarine4	azure	azure1	azure2
azure3	azure4	beige	bisque	bisque1
bisque2	bisque3	bisque4		blanchedalmond
blue	blue1	blue2	blue3	blue4
blueviolet	brown	brown1	brown2	brown3
brown4	burlywood	burlywood1	burlywood2	burlywood3
burlywood4	cadetblue	cadetblue1	cadetblue2	cadetblue3
cadetblue4	chartreuse	chartreuse1	chartreuse2	chartreuse3
chartreuse4	chocolate	chocolate1	chocolate2	chocolate3
chocolate4	coral	coral1	coral2	coral3
coral4	cornflowerblue	cornsilk	cornsilk1	cornsilk2
cornsilk3	cornsilk4	cyan	cyan1	cyan2
cyan3	cyan4	darkblue	darkcyan	darkgoldenrod
darkgoldenrod1	darkgoldenrod2	darkgoldenrod3	darkgoldenrod4	darkgray
darkgreen	darkgrey	darkkhaki	darkmagenta	darkolivegreen
darkolivegreen1	darkolivegreen2	darkolivegreen3	darkolivegreen4	darkorange
darkorange1	darkorange2	darkorange3	darkorange4	darkorchid
darkorchid1	darkorchid2	darkorchid3	darkorchid4	darkred
darksalmon	darkseagreen	darkseagreen1	darkseagreen2	darkseagreen3
darkseagreen4	darkslateblue	darkslategray	darkslategray1	darkslategray2
darkslategray3	darkslategray4	darkslategrey	darkturquoise	darkviolet
deeppink	deeppink1	deeppink2	deeppink3	deeppink4
deepskyblue	deepskyblue1	deepskyblue2	deepskyblue3	deepskyblue4

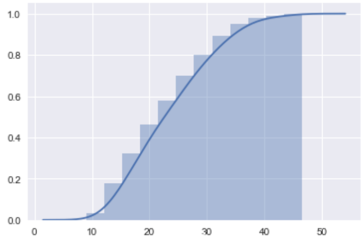
R-colors

Cumulative Frequency Plot

Empirical Cumulative Distribution Function (ECDF):

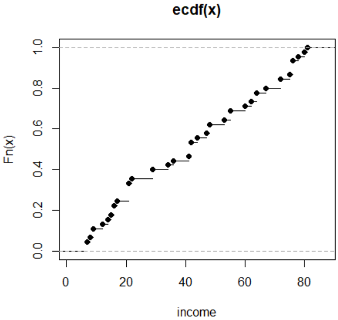
$$\hat{F}(x) = \frac{1}{m}\sum_{i=1}^{m} 1_{\{x_i \le x\}},$$

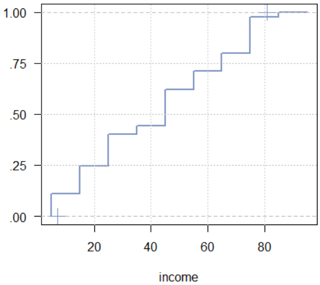
where $1_{\{\cdot\}}$ is the indicator function.



```r
D_c <- Duncan
#left image:
plot.ecdf(D_c$income, xlab = 'income')

#right image:
install.packages("DescTools")
library(DescTools)
PlotECDF(D_c$income, seq(0,100,10), xlab = '
    income')
```

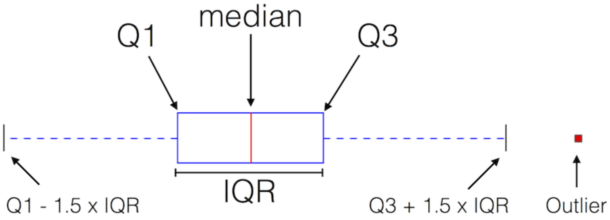




Box Plot

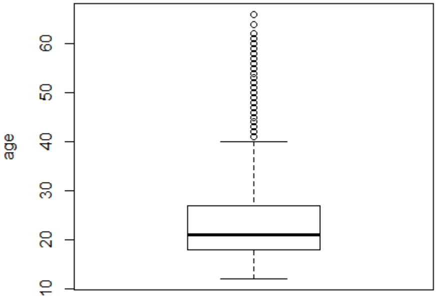
Describes:

- centre of the data,
- spread of the data,
- departure from symmetry,
- identification of outliers of the data



Box Plot

```
D <- Arrests
boxplot(D$age,  ylab = "age")
```



Scatter Plot - Visualization of relations between variables

Idea:
Plot the observations in the x and y diagram
$\longrightarrow$ Relation between x and y becomes apparent

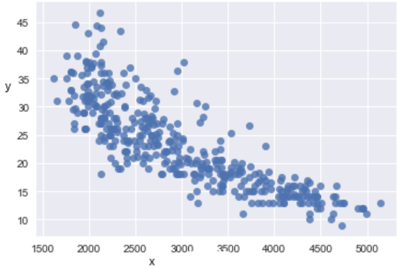
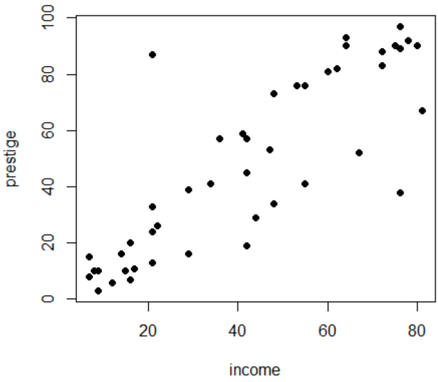


Figure: Scatter plot of two variables x and y.

```
D_c <- Duncan
plot(D_c$income, D_c$prestige, pch=16,xlab='
    income', ylab = 'prestige')
```



Mixing variable types

To get the relation between two variables (*"conditioned"*) one the value of one variable, we can use boxplots.

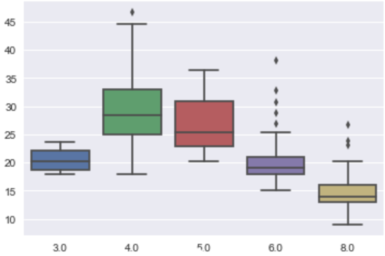
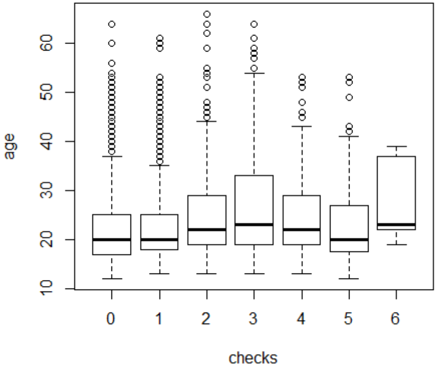
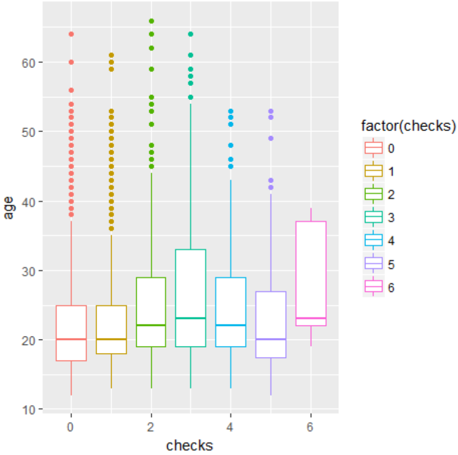


Figure: Box plot by category.

```r
D <- Arrests
boxplot(age~checks, data = D, xlab='checks',
    ylab='age')
```



```
install.packages("ggplot2")
library(ggplot2)
ggplot(D, aes(x=checks, y=age, color=factor(
    checks))) + geom_boxplot()
```



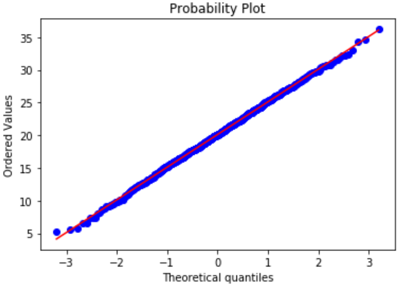
QQ plots

Plots the quantiles of the first data set against the quantiles of the second data set.

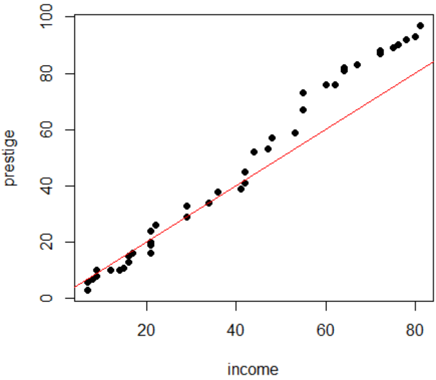
Idea:

- Calculate quantiles of the dataset for x.
- Calculate quantiles of the dataset for y.
- Plot quantiles of x against quantiles of y.

$\Longrightarrow$ If the line is on the 45-degree reference line, the two sets come from a population with the same distribution.

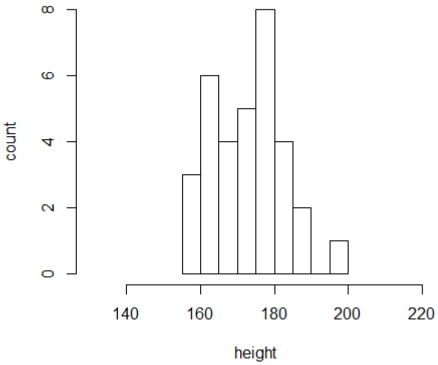


```
D_c <- Duncan
qqplot(D_c$income, D_c$prestige, xlab='income',
    ylab='prestige', pch=16)
abline(0,1,col='red')
```

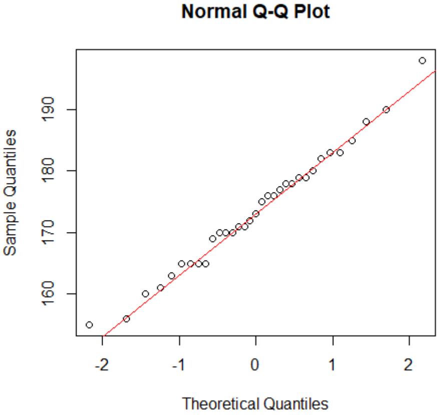


Often QQ-Plots are used to compare sample data to the Normal Distribution.

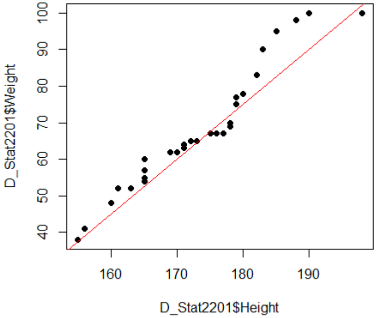
Stat2201 height distribution:



```
library(xlsx)
D_Stat2201 <- read.xlsx("Height_Weight_STAT2201.
    xlsx", 1)
qqnorm(D_Stat2201$Height)
abline(173,10,col='red')
```

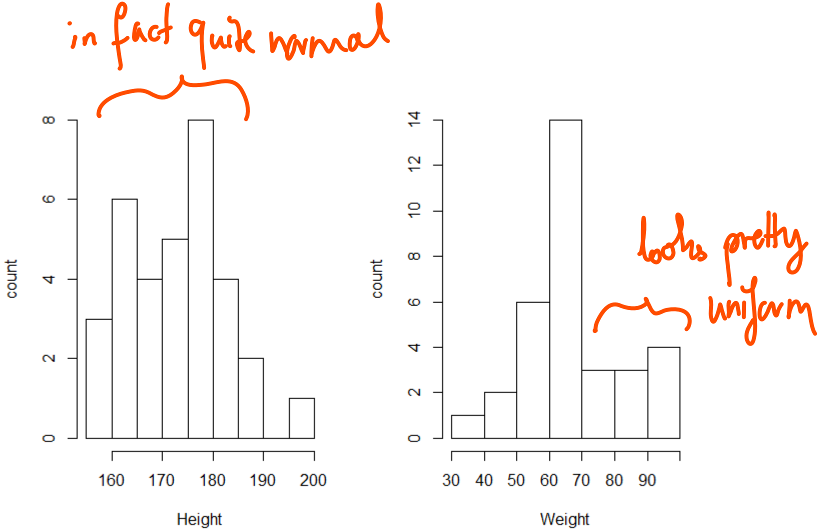


```r
qqplot(D_Stat2201$Height, D_Stat2201$Weight, pch
    =16)
abline(-195,1.5,col='red')
```



⇒ Height & Weight seem to come from the same distrib. but only for a Height ≤ 180

How do you interpret the previous qq plot?



QUIZ - TIME!

Answers, see Word document!

*

Your First Data Analysis

```r
library(carData)
D_Q <- Depredations    #Wolf depredation in 1973
head(Depredations)
```

	longitude	latitude	number	early	late
1	-94.5	46.1	1	0	1
2	-93.0	46.6	2	0	2
3	-94.6	48.5	1	1	0
4	-92.9	46.6	2	0	2
5	-95.9	48.8	1	0	1
6	-92.7	47.1	1	0	1

a) What would be the very first step if someone gives you a dataset?

b) How do you determine the number of observations?

c) Which of the variables are continuous which ones are factors?

d) If you want to investigate the distribution of the latitude with respect to number of depredations, what type of plot (and what R-Code) would you use?

e) What variables do you suspect to be related and how would you test this?

f) Can you think of some other questions you would like to answer with that data set?

Review Chapter 6: Data Description

- Summary Statistics
  a) Sample-Mean,
  b) Sample-Variance,
  c) Sample-Covariance & Sample-Correlation,
  d) Range of Data, Minimum, Maximum,
  e) Median,
  f) P-quantiles.

- Visualization:
  a) Bar-Plot (factor variable),
  b) Pie-Plot (factor variable),
  c) Histogram (continuous variable),
  d) ECDF-Plot,
  e) Box-Plot,
  f) Scatter-Plot (relation of two variables),
  g) QQ-Plot.

Chapter 7–9

- Statistical Inference
- Central Limit Theorem
- Confidence Intervals
- Hypothesis Testing

Statistical inference

Statistical Inference is the process of forming judgements about the parameters.

Assumptions:

- Assume that data $X_1, \ldots, X_n$ is drawn randomly from some **unknown** distribution (identically distributed).
- Assume that the data is independent

  longrightarrow X_i are i.i.d. (independent and identically distributed), i.e.,

  1. $X_i \sim G$ for all $1 \leq i \leq n$
  2. X_is are independent

A statistic

A **statistic** is any function of the observations in a random sample.

$\longrightarrow$ A statistic is itself a R.V.

Examples:

- $g(X_1, X_2, \ldots, X_n) = \overline{X} = \frac{X_1 + X_2 + \cdots + X_n}{n}$ = Sample mean
- $g(X_1, X_2, \ldots, X_n) = \max\{X_1, X_2, \ldots, X_n\}$
- Sample variance and sample standard deviation
- Sample quantiles besides the median, (quartiles and percentiles)

A statistic

- The probability distribution of a statistic is called the **sampling distribution**.
- A **point estimate** of some population parameter θ is a single numerical value $\hat{\theta}$ of a statistic $\hat{\Theta}$.
- The statistic $\hat{\Theta}$ is called the point estimator.

Example:
Sample Mean $= \overline{X} =$ estimator of the population mean, μ.

Normal Distribution - Recap

$X \sim N(\mu, \sigma^2)$ then pdf is

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, \quad x \in \mathbb{R}.$$

- $\mathbb{E}[X] = \mu$ and $\mathrm{Var}(X) = \sigma^2$
- If $\mu = 0$ and $\sigma = 1$ then

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}, \quad x \in \mathbb{R},$$

  = standard normal distribution
- $\frac{X-\mu}{\sigma} \sim \mathrm{N}(0,1)$ = standardization
- $X = \mu + \sigma Z, \quad Z \sim \mathrm{N}(0,1)$

Central Limit Theorem (for sample means)

If $X_1, X_2, \ldots, X_n$ is a random sample of size n taken from a population with mean μ and finite variance σ^2,then

$$\lim_{n\to\infty} \frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}} = Z \sim N(0,1)$$

where $\bar{X}$ is the sample mean. Equivalently,

$$P\left(\frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}} \leq x\right) = \Phi(x)$$

Regardless of X_i's distribution, the sum behaves (approximately) as the Gaussian random variable!

Central Limit Theorem (for sample means)

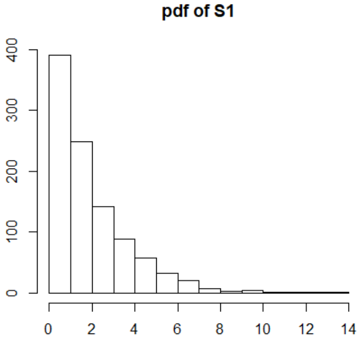
$$\bar{X} \quad \overset{n \to \infty}{\approx} \quad N\left(\mu, \frac{\sigma^2}{n}\right)$$

$S_n = \sum_{i=1}^{n} X_n$ is then distribution

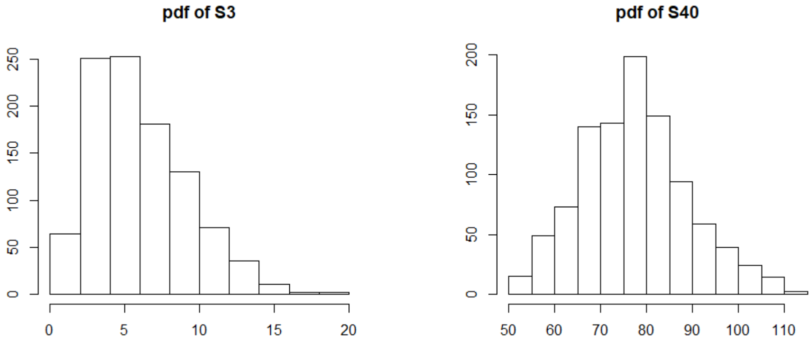
$$S_n \quad \overset{n \to \infty}{\approx} \quad N(n\mu, n\sigma^2)$$

$X_i \sim Exp(0.5)$ (i.i.d.) $\rightarrow S_k = \sum_{i=1}^{k} X_i$

```
M <- matrix(0,50,1000)
M[1,] <- rexp(1000,lambda)
for (i in 2:50){
  M[i,] <- M[i-1,] + rexp(1000, 0.5)
}
```



```
hist(M[3,], main = 'pdf of S3', xlab='', ylab = '')
hist(M[40,], main = 'pdf of S40', xlab='', ylab = '')
```



The standard error of $\overline{X}$

- The standard error of $\overline{X}$ is given by $\frac{\sigma}{\sqrt{n}}$.
- Note that In most practical situations σ is not known but rather estimated.
- The estimated standard error (SE) is:

$$\frac{s}{\sqrt{n}} = \frac{1}{\sqrt{n}}\sqrt{\frac{\sum_{i=1}^{n}(x_i - \overline{x})^2}{n-1}} = \sqrt{\frac{\sum_{i=1}^{n} x_i^2 - n\overline{x}^2}{n(n-1)}}$$

Example:

For a temperature of 100°F and 550 watts, the following measurements of thermal conductivity were obtained:

41.60	41.48	42.34	41.95	41.86
42.18	41.72	42.26	41.81	42.04

$\rightarrow$ sample mean is 41.924

$\rightarrow$ estimated standard error is sample standard deviation s divided by $\sqrt{10}$, here $\frac{0.284}{\sqrt{10}} = 0.0898$

Confidence Interval

confidence interval for μ (the real mean):

$$l \leq \mu \leq u,$$

- Let $X_1, \ldots, X_n$ be collected data
- Endpoints are values of random variables $L = g_1(X_1, \ldots, X_n)$ and $U = g_2(X_1, \ldots, X_n)$ such that

$$P(L(\mathbf{X}) \leq \mu \leq U(\mathbf{X})) = 1 - \alpha, \quad \alpha \in (0, 1).$$

$\longrightarrow$ $1 - \alpha$ is called the **confidence level**.

((l, u) is the $100 \cdot (1 - \alpha)$ % confidence interval.)

Confidence Interval for Mean

Let X_i be i.i.d., then:

- Recall
$$\overline{X} \sim \mathrm{N}\left(\mu, \frac{\sigma^2}{n}\right).$$
- That is, for some positive scalar value $z_{1-\alpha/2}$, we have

$$P\left(\overline{X} \leq \mu + z_{1-\alpha/2}\frac{\sigma}{\sqrt{n}}\right) = P\left(\frac{\overline{X}-\mu}{\frac{\sigma}{\sqrt{n}}} \leq z_{1-\alpha/2}\right) = \Phi(z_{1-\alpha/2})$$

$$P\left(\overline{X} \leq \mu - z_{1-\alpha/2}\frac{\sigma}{\sqrt{n}}\right) = P\left(\frac{\overline{X}-\mu}{\frac{\sigma}{\sqrt{n}}} \leq -z_{1-\alpha/2}\right) = \Phi(-z_{1-\alpha/2}) = 1 - \Phi(z_{1-\alpha/2})$$