This problem set includes problems dealing with most of the topics covered in the course. These include:

- Simple branching processes.
- Simple random walks and gambler's ruin.
- Discrete time Markov chains.
- Discrete time stochastic epidemic models.
- Poisson processes.
- Continuous time Markov chains.
- Continuous time stochastic epidemic models.

Some of the problem refer directly to problems in the textbooks [SP], [MC] and [EM]. The problems also resemble some of the problems that may appear in the final exam.

Note that with the exception of the measure theory questions (which are optional and not for credit), there are 32 questions. The weighting of each such question is 4 points, with the total weighting for the assignment 100 points. Hence you may skip or make mistakes in some questions and still get full marks.

Please hand in a single PDF file for your solutions to the questions. The file may involve a combination of scanned hand written results and typed text/formulas. There should be no real need for computer computations or output. In any case, your submission must have questions answered in order and cannot exceed 8MB in size. Make sure to have your name and student ID on the PDF file, even if you hand in via e-mail.

Note: Introductory measure theoretic probability is optional. If you are interested in exercises on this topic (not for credit), contact the course coordinator.

Questions:

1. Simple branching processes:
   (a) Consider a simple branching process S_n with $S_0 = 1$ and Bernoulli-distributed offspring distribution with mean p. Determine the probability of ultimate extinction as a function of p.
   (b) At the start of the course, we considered examples of population models that "double every time period". The basic example was $X(t+1) = 2X(t)$ (deterministic). Then we considered simple branching processes. Refer now to Section 4.2.1 in [SP-4] and find parameters of a continuous time Branching process that "doubles every time period".
2. Simple random walks and gambler's ruin:
   (a) Consider a random walk $S_n = \sum_{i=1}^{n} X_i$ where X_i are i.i.d. random variables distributed uniformly over the set $\{-1, 0, 1\}$. Derive an expression for $\text{Cov}(S_n, S_{n+k})$ with k being a positive integer.
   (b) Question 1.3.2, pg 18, [MC-1].
3. Discrete time Markov chains:
   (a) Question 1.20, pg 81 [SP-1].
   (b) Question 1.33, pg 83-84 [SP-1].
   (c) Question 1.48, pg 88 [SP-1].
   (d) Question 1.57, pg 90 [SP-1].
   (e) Question 1.65, pg 92 [SP-1].
   (f) Question 1.73, pg 94 [SP-1].
   (g) Question 1.5.4, pg 29 [MC-1].
   (h) Question 1.8.4, pg 46 [MC-1].
   (i) Consider Theorem 1.8.3 on pg 41 [MC-1]. Rewrite the theorem and the proof specializing to two state Markov chains on the state space $\{1, 2\}$.
4. Discrete time stochastic epidemic models:
   (a) Derive the chain binomial representation in Equation (1.4.2) of pg 12 [EM-1].
   (b) Derive Equations (4.1.4) pg 107 [EM-4]. This was part of Project 1 as well.
5. Poisson processes:
   (a) Question 2.23, pg 119 [SP-2].
   (b) Question 2.30, pg 120 [SP-2].
   (c) Question 2.43, pg 122 [SP-2].
   (d) Question 2.47, pg 123 [SP-2].
   (e) Consider the algorithm for generating a Poisson random variable with mean λ using Equation (3.15) pg 100 [SWJ-3]. Explain (prove) why this algorithm works.
   (f) Consider Theorem 2.4.6 pg 79 [MC-2]. Rewrite the theorem and the proof, specializing to the case of $n = 2$. Plot/sketch the joint distribution of J_1 and J_2 in this case, with $X_{10} = 2$.
6. Continuous time Markov chains:

(a) Question 4.7, pg 193 [SP-4].

(b) Question 4.14, pg 194 [SP-4].

(c) Question 4.20, pg 195 [SP-4].

(d) Consider the M/M/∞ queue with arrival rate λ and service rate μ. Determine the variance of the stationary distribution of the number of customers in the queue.

(e) Question 3.6.3, pg 123 [MC-3].

(f) Consider the two state continuous time Markov chain with birth rate λ and death rate μ. Denote the state space $\{0, 1\}$. Write the forward equations and the backward equations and use one of these to find expressions for $\mathbb{P}(X_t = 1|X_0 = 0)$.

(g) Consider the M/M/1/K queue where finite K is the total system capacity (state space is $\{0, 1, 2, \ldots, K\}$). Assume $\lambda \neq \mu$. Find an expression for the mean number of customers in the system in steady state.

(h) For the M/M/1/K Try to reproduce the computations for $\mathbb{P}(W > x)$ as in pg 335 [SWJ-10] to obtain some expression for the CCDF of the waiting time.

(i) Consider a 3 state continuous time Markov chain as in Listing 10.6 pg 327 [SWJ-10]. Diagonalize the generator matrix Q for computing the matrix exponential by hand. Use this to obtain the distribution of the chain at time $T = 0.25$ and obtain results that agree with the output of Listing 10.6.

7. Continuous time stochastic epidemic models:

(a) Question 3.2, pg 101 [EM-3].

(b) Consider the SIS epidemic as in Project 2 with $a > 0$ using the notation of Project 2. Find an expression for stationary distribution when $N = 2$ (state space is $\{0, 1, 2\}$).