STAT3004/STAT7304 Sem1, 2020
Problem Set Solution [redacted]

Good solutions taken from students

1a:

(a) Consider a simple branching process S_n with $S_0 = 1$ and Bernoulli-distributed offspring distribution with mean p. Determine the probability of ultimate extinction as a function of p.

In lectures we stated as a theorem that the probability of ultimate extinction is given by the minimal non-negative solution to $\eta = G_X(\eta)$, where G_X is the PGF of offspring distribution for the branching process S_n. First we note the PGF of the offspring distribution $G_X(z)$, given that $X_{i,n} \sim^{iid} \mathsf{Ber}(p)$.

$$
\begin{aligned}
G_X(z) = \mathbb{E}z^X &= \sum_{k=0}^{\infty} z^k \mathbb{P}(X = k) \\
&= z^0(1-p) + z^1 p \\
&= 1 - p + pz \\
\eta = G_X(\eta) &= 1 - p + p\eta \\
\implies \eta(1-p) &= 1 - p \implies \eta = 1 \text{ or } p = 1
\end{aligned}
$$

If $p = 1$, we have free choice for η and so $\eta = 0$ is the minimal non-negative solution. So we have that the probability of extinction is

$$
\eta = \begin{cases} 1 \text{ if } 0 \leq p < 1 \\ 0 \text{ if } p = 1 \end{cases}
$$

1b

(b) At the start of the course, we considered examples of population models that "double every time period". The basic example was $X(t+1) = 2X(t)$ (deterministic). Then we considered simple branching processes. Refer now to Section 4.2.1 in [SP-4] and find parameters of a continuous time Branching process that "doubles every time period".

Recall the *Yule process* from Example 4.11, where we have Y_t and each particle splits into two at rate β, so $q(i, i+1) = \beta i$ is the only transition. We can use the theorem regarding the Yule process, theorem 4.2 in the text:

Theorem. *The transition probability of the Yule process is given by*

$$p_t(1,j) = e^{-\beta t}(1-e^{-\beta t})^{j-1}, \quad j \geq 1$$
$$p_t(i,j) = \binom{j-1}{i-1}(e^{-\beta t})^i(1-e^{-\beta t})^{j-i}.$$

That is, $p_t(1,j)$ is a geometric distribution with success probability $e^{-\beta t}$ and hence mean $e^{\beta t}$.

The key thing to note from the above theorem is that the mean $\mathbb{E}Y_t = e^{\beta t}$. A "doubling" branching process has expectation $\mathbb{E}Y_t = 2^t$, so we can solve for the parameter β quite easily:

$$\mathbb{E}Y_t = 2^t \implies e^{\beta t} = 2^t \implies \beta t = \log(2^t) = t\log 2 \implies \boxed{\beta = \log 2}.$$

Or

Assume that we are in a continuous time Branching process, take Y_t to be the continuous Markov Chain with jump rate given by $q(i, i+1) = \beta i$.

"Doubles every time period" in this process is equivalent to doubling the expected value of Y_t after one time unit, which means:

$$E(Y_{t+1}) = 2E(Y_t) \ *$$

Now, note that Y_t satisfies $\frac{d}{dt}E(Y_t) = \beta E(Y_t)$, which implies $E(Y_t) = e^{\beta t}$ **.

Combining * and ** together, we get:

$$e^{\beta(t+1)} = 2e^{\beta t}$$
$$\frac{e^{\beta(t+1)}}{e^{\beta t}} = 2$$
$$e^{\beta t + \beta - \beta t} = 2$$
$$e^{\beta} = 2$$
$$\beta = ln(2) \approx 0.6931$$

Thus, by taking a continuous time Markov chain Y_t to be the continuous Markov Chain with jump rate given by $q(i, i+1) = \beta i$, where $\beta = ln(2) \approx 0.6931$, we have a continuous time Branching process that the expected value of Y doubles every time t increases by one.

Or

We have $q(i, i+1) = \lambda_i$, $q(i, i-1) = \mu_i$. Using the notation of [SP], let $z(t)$ denote the process with $z(0) = 1$. We want the mean to double every time period. First note that we can consider the rate of change of expectation:

$$\frac{d}{dt}\mathbb{E}z(t) = (\lambda - \mu)\mathbb{E}z(t)$$

$$\begin{aligned} &\implies \mathbb{E}z(t) = Ae^{(\lambda-\mu)t} \\ &\quad z(0) = 1 \implies \mathbb{E}(z(0) = 1 \\ &\implies \mathbb{E}z(t) = e^{(\lambda-\mu)t} \end{aligned}$$

We want the condition that

$$\begin{aligned} &\mathbb{E}z(t+1) = 2\mathbb{E}z(t) \\ \iff &e^{(\lambda-\mu)(t+1)} = 2e^{(\lambda+\mu)(t+1)} \\ \iff &e^{(\lambda-\mu)} = 2 \implies \lambda + \mu = \ln 2 \end{aligned}$$

This condition will yield a cts time branching process which doubles every time increment.

2a

(a) Consider a random walk $S_n = \sum_{i=1}^{n} X_i$ where X_i are i.i.d. random variables distributed uniformly over the set $\{-1, 0, 1\}$. Derive an expression for $\mathrm{Cov}(S_n, S_{n+k})$ with k being a positive integer.

Question 2 :

a) $S_n = \sum_{k=1}^{n} X_i$, X_i iid and unif $\{-1, 0, 1\}$

$E(X) = 0$

Let $k \in \mathbb{N}^*$

$Var(X) = \frac{2}{3}$

$$Cov(S_n, S_{n+k}) = E(S_n \cdot S_{n+k}) - \underbrace{E(S_n) \cdot E(S_{n+k})}_{0}$$

$$= E(S_n^2) + E\left(\sum_{i=1}^{n} X_i \cdot \sum_{j=n+1}^{n+k} X_j\right)$$

$$= Var(S_n) + \sum_{i=1}^{n} \sum_{j=n+1}^{n+k} E(X_i \cdot X_j)$$

because $i \neq j$
$\Rightarrow X_i, X_j$ independent
$\Rightarrow E(X_i) E(X_j) = E(X_i X_j)$
and $E(X) = 0$

$$= Var(S_n) = n Var(X) = \frac{2n}{3}$$

2b Question 1.3.2, pg 18, [MC-1].

1.3.2 A gambler has £2 and needs to increase it to £10 in a hurry. He can play a game with the following rules: a fair coin is tossed; if a player bets on the right side, he wins a sum equal to his stake, and his stake is returned; otherwise he loses his stake. The gambler decides to use a bold strategy in which he stakes all his money if he has £5 or less, and otherwise stakes just enough to increase his capital, if he wins, to £10.

Let $X_0 = 2$ and let X_n be his capital after n throws. Prove that the gambler will achieve his aim with probability 1/5.

What is the expected number of tosses until the gambler either achieves his aim or loses his capital?

Since we are dealing with doubling, with a potential maximum amount of £10, we can define the state space to be the set of even numbers up to 10, as well as 0, being $\Omega = \{0, 2, 4, 6, 8, 10\}$. This allows us to construct the following Markov chain based on the cases:

$$P = \begin{array}{c} 0 \\ 2 \\ 4 \\ 6 \\ 8 \\ 10 \end{array} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{array}{l} \text{can't get any more money} \\ \text{double or nothing (to £4)} \\ \text{double or nothing (to £8)} \\ \text{bets just enough (£4) to get to £10} \\ \text{bets just enough (£2) to get to £10} \\ \text{stops gambling} \end{array}$$

Define $h_i = \mathbb{P}_i(\text{hit } 10)$ and $k_i = \mathbb{E}_i[\text{time to hit } \{0, 10\}]$. We wish to find h_2. Note that $h_0 = 0$ (since we would be stuck with £0 if we have £0) and $h_{10} = 1$ (we have reached our goal). We can express the other hitting times with the following equations:

$$\begin{aligned} h_2 &= \frac{1}{2}h_4 + \frac{1}{2}h_0 \\ h_4 &= \frac{1}{2}h_8 + \frac{1}{2}h_0 \\ h_6 &= \frac{1}{2}h_2 + \frac{1}{2}h_{10} \\ h_8 &= \frac{1}{2}h_6 + \frac{1}{2}h_{10} \end{aligned} \implies \begin{aligned} h_2 &= \frac{1}{2}h_4 \\ h_4 &= \frac{1}{2}h_8 \\ h_6 &= \frac{1}{2}h_2 + \frac{1}{2} \\ h_8 &= \frac{1}{2}h_6 + \frac{1}{2} \end{aligned}$$

Solving backwards, we have

$$\begin{aligned} h_2 = \frac{1}{2}h_4 &= \frac{1}{2}\left(\frac{1}{2}h_8\right) \\ &= \frac{1}{2}\left(\frac{1}{2}\left(\frac{1}{2}h_6 + \frac{1}{2}\right)\right) \\ &= \frac{1}{2}\left(\frac{1}{2}\left(\frac{1}{2}\left(\frac{1}{2}h_2 + \frac{1}{2}\right) + \frac{1}{2}\right)\right) \\ &= \frac{1}{2}\left(\frac{1}{2}\left(\frac{1}{4}h_2 + \frac{1}{4} + \frac{1}{2}\right)\right) \\ &= \frac{1}{2}\left(\frac{1}{8}h_2 + \frac{1}{8} + \frac{1}{4}\right) \\ &= \frac{1}{16}h_2 + \frac{1}{16} + \frac{1}{8} \\ \implies \frac{15}{16}h_2 = \frac{3}{16} &\implies \boxed{h_2 = \frac{1}{5}}. \end{aligned}$$

To find k_2, note that $k_0 = k_{10} = 0$. We then have

$$\begin{aligned} k_2 &= \frac{1}{2}k_0 + \frac{1}{2}k_4 + 1 \\ k_4 &= \frac{1}{2}k_0 + \frac{1}{2}k_8 + 1 \\ k_6 &= \frac{1}{2}k_2 + \frac{1}{2}k_{10} + 1 \\ k_8 &= \frac{1}{2}k_6 + \frac{1}{2}k_{10} + 1 \end{aligned} \implies \begin{aligned} k_2 &= \frac{1}{2}k_4 + 1 \\ k_4 &= \frac{1}{2}k_8 + 1 \\ k_6 &= \frac{1}{2}k_2 + 1 \\ k_8 &= \frac{1}{2}k_6 + 1 \end{aligned}$$

Combining the equations together in terms of k_2, we obtain

$$\begin{aligned} k_2 &= \frac{1}{2}\left(\frac{1}{2}k_8 + 1\right) + 1 \\ &= \frac{1}{2}\left(\frac{1}{2}\left(\frac{1}{2}k_6 + 1\right) + 1\right) + 1 \\ &= \frac{1}{2}\left(\frac{1}{2}\left(\frac{1}{2}\left(\frac{1}{2} + k_2 + 1\right) + 1\right) + 1\right) + 1 \\ &= \frac{1}{2}\left(\frac{1}{2}\left(\frac{1}{4}k_2 + \frac{1}{2} + 1\right) + 1\right) + 1 \\ &= \frac{1}{16}k_2 + \frac{1}{8} + \frac{1}{4} + \frac{1}{2} + 1 \\ \implies \frac{15}{16}k_2 &= \frac{1}{8} + \frac{1}{4} + \frac{1}{2} + 1 = \frac{15}{16} \implies \boxed{k_2 = 2}. \end{aligned}$$

Therefore, starting with £2, the gambler will achieve his aim with probability $\boxed{1/5}$ and is expected to throw $\boxed{\text{2 tosses}}$ to achieve either his aim or lose his capital.

3a.
Question 1.20, pg 81 [SP-1].

1.20. Three of every four trucks on the road are followed by a car, while only one of every five cars is followed by a truck. What fraction of vehicles on the road are trucks?

This can be thought of as a Markov chain. If the current vehicle in the chain is a truck, the probability of the next vehicle being a car is 3/4. If the current vehicle is a car, the probability that the next vehicle is a truck is 1/5. A transition probability can be deduced from this information, such that

$$P = \begin{array}{c|cc} & T & C \\ \hline T & 1/4 & 3/4 \\ C & 1/5 & 4/5 \end{array}$$

To find what fraction of vehicles on the road are trucks, we can find the stationary distribution, ie.

$$\pi = [\pi_T \quad \pi_C] \text{ s.t. } \pi = \pi P \text{ and } \pi_T + \pi_C = 1.$$

$$[\pi_T \quad \pi_C] = [\pi_T \quad \pi_C]\begin{bmatrix} 1/4 & 3/4 \\ 1/5 & 4/5 \end{bmatrix}$$

$\Rightarrow$ (1) $\pi_T = \frac{1}{4}\pi_T + \frac{1}{5}\pi_C$

(2) $\pi_C = \frac{3}{4}\pi_T + \frac{4}{5}\pi_C$

(1) can be rewritten as

$$\pi_T - \tfrac{1}{4}\pi_T = \tfrac{1}{5}\pi_C$$

$$\frac{3}{4}\pi_T = \frac{1}{5}\pi_C$$

$$\pi_T = \frac{4}{3} \times \frac{1}{5}\pi_C$$

$$\pi_T = \frac{4}{15}\pi_C$$

We know that $\pi_T + \pi_C = 1$, so $\pi_C + \frac{4}{15}\pi_C = 1$

$$\Rightarrow \pi_C\left(\frac{19}{15}\right) = 1$$

$$\Rightarrow \pi_C = \frac{15}{19}$$

$$\Rightarrow \pi_T = \frac{4}{19}$$

So $\frac{4}{19}$ of the vehicles on the road are trucks.

3b.

Question 1.33, pg 83-84 [SP-1]

1.33. In a large metropolitan area, commuters either drive alone (A), carpool (C), or take public transportation (T). A study showed that transportation changes according

to the following matrix:

	A	**C**	**T**
A	.8	.15	.05
C	.05	.9	.05
S	.05	.1	.85

In the long run what fraction of commuters will use the three types of transportation?

Since we are interesting in the long-run fractions of the population which use each type of transport we are searching for the stationary distribution. The stationary distribution $\pi = (\pi_A, \pi_C, \pi_T)$ must satisfy the following matrix expression by definition:

$$\pi P = \pi \Rightarrow \pi(P - I) = 0$$

Where P is the given matrix of transition probabilities, I is the 3×3 identity matrix and 0 is a 1×3 zeros vector:

$$P = \begin{pmatrix} 0.8 & 0.15 & 0.05 \\ 0.05 & 0.9 & 0.05 \\ 0.05 & 0.1 & 0.85 \end{pmatrix}$$

We can solve this by noting that we have a fourth equation the distribution must satisfy:

$$\pi_A + \pi_C + \pi_T = 1$$

Thus we can replace the third column of $(p - I)$ with all ones and replace the third element of the zeros vector with a one:

$$\pi \begin{pmatrix} -0.2 & 0.15 & 1 \\ 0.05 & -0.1 & 1 \\ 0.05 & 0.1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}$$

So we get:

$$\pi = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -0.2 & 0.15 & 1 \\ 0.05 & -0.1 & 1 \\ 0.05 & 0.1 & 1 \end{pmatrix}^{-1}$$

Performing this matrix inverse calculation using MATLAB we get:

$$\pi = \begin{pmatrix} 0.2 & 0.55 & 0.25 \end{pmatrix}$$

Which are the long-run fractions of commuters who drive alone, carpool or take public transportation, respectively.

3c.

Question 1.48, pg 88 [SP-1].

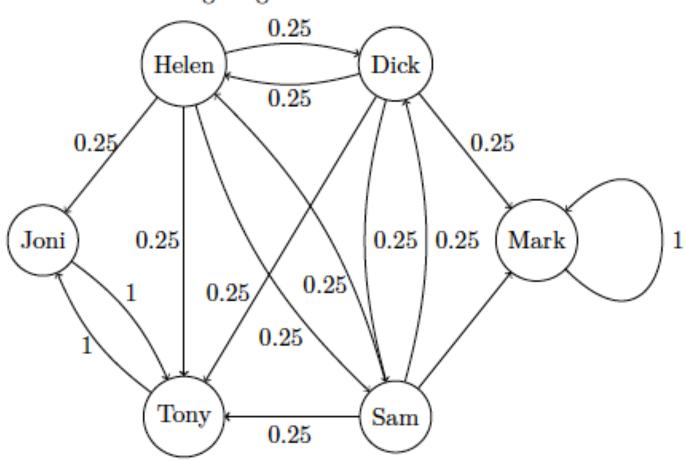
1.48. Six children (Dick, Helen, Joni, Mark, Sam, and Tony) play catch. If Dick has the ball, he is equally likely to throw it to Helen, Mark, Sam, and Tony. If Helen has the ball, she is equally likely to throw it to Dick, Joni, Sam, and Tony. If Sam has the ball, he is equally likely to throw it to Dick, Helen, Mark, and Tony. If either Joni or Tony gets the ball, they keep throwing it to each other. If Mark gets the ball, he runs away with it. (a) Find the transition probability and classify the states of the chain. (b) Suppose Dick has the ball at the beginning of the game. What is the probability Mark will end up with it?

(a)

Let the states of the chain be $\{D, H, J, M, S, T\}$ corresponding to Dick, Helen, Joni, Mark, Sam and Tony respectively. The transition probability is the following:

D	0	0.25	0	0.25	0.25	0.25	equally likely to throw to Helen, Mark, Sam, Tony
H	0.25	0	0.25	0	0.25	0.25	equally likely to throw to Dick, Joni, Sam, Tony
J	0	0	0	0	0	1	always throws it to Tony
M	0	0	0	1	0	0	runs away with the ball
S	0.25	0.25	0	0.25	0	0.25	equally likely to throw to Dick, Helen, Mark, Tony
T	0	0	1	0	0	0	always throws it to Joni

This can be represented with the following diagram:



Note that the <u>communicating classes</u> of this Markov Chain are

$$\{M\}, \{T, J\} \text{ and } \{D, H, S\}.$$

Since M is a closed, irreducible set, M is RECURRENT. $\{T, J\}$ is also a closed, irreducible set, so T and J are both RECURRENT.

For state D, notice that:

$$P_{DH} = \frac{1}{4} > 0, \text{ but } P_{HD} = \frac{1}{4} < 1, \text{ so } D \text{ is TRANSIENT}$$

For state H,

$$P_{HD} = \frac{1}{4} > 0, \text{ but } P_{DH} = \frac{1}{4} < 1 \text{ so } H \text{ is TRANSIENT}$$

For state S,

$$P_{SD} = \frac{1}{4} > 0, \text{ but } P_{DS} = \frac{1}{4} < 1, \text{ so } S \text{ is TRANSIENT}$$

[redacted]. If we now let $h(x)$ be the probability that the ball gets to Mark if person x has the ball, then $h(\text{Joni}) = h(\text{Tony}) = 0$ and $h(\text{Mark}) = 1$. Then, we know that

$$h(x) = \sum_y p(x,y)h(y)$$

and by **Theorem 1.27**[SP, ED-2] that this will also be the probability of the ball reaching Mark, before Joni and Tony. Writing out the system of equations we get

$$\begin{aligned}
h(\textbf{Dick}) =& \frac{1}{4}h(\textbf{Helen}) + \frac{1}{4}h(\textbf{Mark}) + \frac{1}{4}h(\textbf{Sam}) + \frac{1}{4}h(\textbf{Tony}) \\
h(\textbf{Helen}) =& \frac{1}{4}h(\textbf{Dick}) + \frac{1}{4}h(\textbf{Joni}) + \frac{1}{4}h(\textbf{Sam}) + \frac{1}{4}h(\textbf{Tony}) \\
h(\textbf{Joni}) =& h(\textbf{Tony}) \\
h(\textbf{Sam}) =& \frac{1}{4}h(\textbf{Dick}) + \frac{1}{4}h(\textbf{Helen}) + \frac{1}{4}h(\textbf{Mark}) + \frac{1}{4}h(\textbf{Tony}) \\
h(\textbf{Tony}) =& h(\textbf{Joni})
\end{aligned}$$

Simplifying this

$$\begin{aligned}
h_D =& \frac{1}{4}h_H + \frac{1}{4} + \frac{1}{4}h_S \\
h_H =& \frac{1}{4}h_D + \frac{1}{4}h_s \\
h_s =& \frac{1}{4}h_D + \frac{1}{4}h_H + \frac{1}{4}
\end{aligned}$$

Subtracting the last equation from the first we get

$$h_D - h_s = \frac{1}{4}h_s - \frac{1}{4}h_D \implies h_D = h_S$$

h_H is then

$$h_H = \frac{1}{2}h_D$$

Substituting these results into the first equation we get

$$h_D = \frac{1}{8}h_d + \frac{1}{4} + \frac{1}{4}h_d \implies \frac{5}{8}h_d = \frac{1}{4} \implies h_d = \frac{2}{5}$$

Therefore, the probability that Mark will get the Ball if it starts with Dick, is 2/5.

3d.

Question 1.57, pg 90 [SP-1].

1.57. 3. Two barbers and two chairs. Consider the following chain

	0	**1**	**2**	**3**	**4**
0	0	1	0	0	0
1	0.6	0	0.4	0	0
2	0	0.75	0	0.25	0
3	0	0	0.75	0	0.25
4	0	0	0	0.75	0.25

(a) Find the stationary distribution. (b) Compute $P_x(V_0 < V_4)$ for $x = 1, 2, 3$. (c) Let $\tau = \min\{V_0, V_4\}$. Find $E_x\tau$ for $x = 1, 2, 3$.

The stationary distribution can be computed from the system of equations

$$\pi P = \pi \quad \sum_i \pi_i = 1$$

In this case we get the system of equations

$$\begin{aligned} \frac{3}{5}\pi_1 &= \pi_0 \\ \pi_0 + \frac{3}{4}\pi_2 &= \pi_1 \\ \frac{2}{5}\pi_1 + \frac{3}{4}\pi_3 &= \pi_2 \\ \frac{1}{4}\pi_2 + \frac{3}{4}\pi_4 &= \pi_3 \\ \frac{1}{4}\pi_3 + \frac{1}{4}\pi_4 &= \pi_4 \\ \pi_0 + \pi_1 + \pi_2 + \pi_3 + \pi_4 &= 1 \end{aligned}$$

Which in matrix form $P^T\pi^T = \pi^T$, with the second last equation replaced with the normalising equation, is

$$\begin{pmatrix} -1 & \frac{3}{5} & 0 & 0 & 0 \\ 1 & -1 & \frac{3}{4} & 0 & 0 \\ 0 & \frac{2}{5} & -1 & \frac{3}{4} & 0 \\ 0 & 0 & \frac{1}{4} & -1 & \frac{3}{4} \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \pi_0 \\ \pi_1 \\ \pi_2 \\ \pi_3 \\ \pi_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Which once reduced gives a stationary distribution of

$$\begin{pmatrix} \pi_0 \\ \pi_1 \\ \pi_2 \\ \pi_3 \\ \pi_4 \end{pmatrix} = \begin{pmatrix} 81/320 \\ 27/64 \\ 9/40 \\ 3/40 \\ 1/40 \end{pmatrix}$$

The notation $\mathbb{P}_x(V_0 < V_4)$ is equivalent to "reaching 0 before 4", or that the "hitting time of 0 is less than the hitting time of 4". Note that $\mathbb{P}_0(V_0 < V_4) = 1$ and $\mathbb{P}_4(V_0 < V_4) = 0$, since if we have already reached 0 or 4 then the chance of us reaching 0 first is 1 and 0 respectively. We wish to solve the following:

$$\begin{aligned}
\mathbb{P}_1(V_0 < V_4) &= 0.6\mathbb{P}_0(V_0 < V_4) + 0.4\mathbb{P}_2(V_0 < V_4) = 0.6 + 0.4\mathbb{P}_2(V_0 < V_4) \\
\mathbb{P}_2(V_0 < V_4) &= 0.75\mathbb{P}_1(V_0 < V_4) + 0.25\mathbb{P}_3(V_0 < V_4) \\
\mathbb{P}_3(V_0 < V_4) &= 0.75\mathbb{P}_2(V_0 < V_4) + 0.25\mathbb{P}_4(V_0 < V_4) = 0.75\mathbb{P}_2(V_0 < V_4)
\end{aligned}$$

This system of equations can be solved with the matrix

$$\begin{pmatrix} -1 & 0.4 & 0 \\ 0.75 & -1 & 0.25 \\ 0 & 0.75 & -1 \end{pmatrix}^{-1} \begin{pmatrix} -0.6 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 39/41 \\ 36/41 \\ 27/41 \end{pmatrix}.$$

This gives us

$$\boxed{\mathbb{P}_1(V_0 < V_4) = 39/41, \quad \mathbb{P}_2(V_0 < V_4) = 36/41, \quad \mathbb{P}_3(V_0 < V_4) = 29/41}.$$

c) τ can equivalently be expressed as: $\tau = \inf\{n \geq 0 X_n \in \{0,4\}\}$, where X_n is the state of the chain at step n. In this case we can define $C = \{1,2,3\}$ as the state space minus $\{0,4\}$, and we have that C is finite and the chain is such that the probability of the chain entering $\{0,4\}$ in a finite number of steps when it starts in any state $x \in C$ is non-zero. Thus this problem fulfils the requirements of Theorem 1.29 in SP-1 and we get:

$$g(x) = 1 + \sum_y p_{xy} g(y) = E_x\tau \forall x \in \{1,2,3\}$$

$g(0) = g(4) = 0$ by definition. As such:

$$\begin{aligned}
g(1) &= 1 + 0.4g(2) \\
g(2) &= 1 + 0.75g(1) + 0.25g(3) \\
g(3) &= 1 + 0.75g(2)
\end{aligned}$$

$$\Rightarrow \begin{pmatrix} g(1) \\ g(2) \\ g(3) \end{pmatrix} = \begin{pmatrix} 1 & -0.4 & 0 \\ -0.75 & 1 & -0.25 \\ 0 & -0.75 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 105/41 \\ 160/41 \\ 161/41 \end{pmatrix}$$

These are $\mathbb{E}_x\tau$ for $x = 1,2,3$, respectively.

3e.
Question 1.65, pg 92 [SP-1].

1.65. *Ehrenfest chain.* Consider the Ehrenfest chain, Example 1.2, with transition probability $p(i, i+1) = (N-i)/N$, and $p(i, i-1) = i/N$ for $0 \le i \le N$. Let $\mu_n = E_x X_n$. (a) Show that $\mu_{n+1} = 1 + (1 - 2/N)\mu_n$. (b) Use this and induction to conclude that

$$\mu_n = \frac{N}{2} + \left(1 - \frac{2}{N}\right)^n (x - N/2)$$

From this we see that the mean μ_n converges exponentially rapidly to the equilibrium value of $N/2$ with the error at time n being $(1-2/N)^n(x - N/2)$.

(a)

Proof. Note that we can use the tower property to obtain

$$\mathbb{E}_x[X_{n+1}] = \mathbb{E}_x[\mathbb{E}_x[X_{n+1} \mid X_n]].$$

We can now calculate $\mathbb{E}_x[X_{n+1} \mid X_n]$. Note that this conditional expectation relies on our current state X_n. From X_n, we can move $+1$ with probability $\frac{N-i}{N} = \frac{N-X_n}{N}$, and move -1 with probability $\frac{i}{N} = \frac{X_n}{N}$. Since the expectation is $\sum$ state $\times$ probability, we have

$$\mathbb{E}_x[X_{n+1} \mid X_n] = \frac{N - X_n}{N} - \frac{X_n}{N} + X_n = 1 - \frac{2X_n}{N} + X_n.$$

We can now take the expectation of this whole equation:

$$\mathbb{E}_x[E_x[X_{n+1} \mid X_n]] = \mathbb{E}_x\left[1 - \frac{2X_n}{N} + X_n\right] = \mathbb{E}_x[1] - \frac{2}{N}\mathbb{E}_x[X_n] + \mathbb{E}_x[X_n]$$
$$= 1 - \frac{2}{N}\mu_n + \mu_n = 1 + \left(1 - \frac{2}{N}\right)\mu_n$$

(b)

Proof. We assume that $\mu_n = \frac{N}{2} + (1 - \frac{2}{N})^n(x - N/2)$ and we wish to prove μ_{n+1} holds.

$$\mu_{n+1} = 1 + \left(1 - \frac{2}{N}\right)\mu_n = 1 + \left(1 - \frac{2}{N}\right)\left(\frac{N}{2} + \left(1 - \frac{2}{N}\right)^n\left(x - \frac{N}{2}\right)\right)$$
$$= \cancel{1} + \frac{N}{2} + \left(1 - \frac{2}{N}\right)^n\left(x - \frac{N}{2}\right) - \cancel{1} - \frac{2}{N}\left(1 - \frac{2}{N}\right)^n\left(x - \frac{N}{2}\right)$$
$$= \frac{N}{2} + \left(1 - \frac{2}{N}\right)^n\left(x - \frac{N}{2}\right) - \frac{2}{N}\left(1 - \frac{2}{N}\right)^n\left(x - \frac{N}{2}\right)$$
$$= \frac{N}{2} + \left(1 - \frac{2}{N}\right)\left(1 - \frac{2}{N}\right)^n\left(x - \frac{N}{2}\right) = \frac{N}{2} + \left(1 - \frac{2}{N}\right)^{n+1}\left(x - \frac{N}{2}\right)$$

which by induction allows us to conclude that $\mu_n = \frac{N}{2} + \left(1 - \frac{2}{N}\right)^n (x - N/2)$.

3f.

Question 1.73, pg 94 [SP-1].

1.73. Consider the Markov chain with state space $\{1, 2, \ldots\}$ and transition probability

$$p(m, m+1) = m/(2m+2) \qquad \text{for } m \geq 1$$
$$p(m, m-1) = 1/2 \qquad \text{for } m \geq 2$$
$$p(m, m) = 1/(2m+2) \qquad \text{for } m \geq 2$$

and $p(1,1) = 1 - p(1,2) = 3/4$. Show that there is no stationary distribution.

The general form of the transition matrix for this Markov Chain is

$$P = \begin{pmatrix} \frac{3}{4} & \frac{1}{4} & 0 & 0 & 0 & \ldots \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{3} & 0 & 0 & \ldots \\ 0 & \frac{1}{2} & \frac{1}{8} & \frac{3}{8} & 0 & \ldots \\ 0 & 0 & \frac{1}{2} & \frac{1}{10} & \frac{2}{5} & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

We know that π is a stationary distribution if the global balance equations hold

$$\pi(x) = \sum_y p(y, x)\pi(y)$$

We will now prove by induction that for this Markov chain this implies that, if it has a stationary distribution, then $\pi_i = \frac{1}{i}\pi_1$, $i \in \mathbb{N}$. Checking the base case for $i = 2$ we see from the first column of our transition matrix P that

$$\pi_1 = \frac{3}{4}\pi_1 + \frac{1}{2}\pi_2 \implies \pi_1 = 2\pi_2 \implies \pi_2 = \frac{1}{2}\pi_1$$

Which proves our base case. Suppose that this is true up to some integer $k \geqslant 1$. Then consider π_{k+1}

$$\begin{aligned} \pi_k =& \frac{k-1}{2(k-1)+2}\pi_{k-1} + \frac{1}{2k+2}\pi_k + \frac{1}{2}\pi_{k+1} \\ \implies \frac{1}{k}\pi_1 =& \frac{k-1}{2k}\left(\frac{1}{k-1}\right)\pi_1 + \frac{1}{2k+2}\left(\frac{1}{k}\right)\pi_1 + \frac{1}{2}\pi_{k+1} \quad \text{Inductive Hypothesis} \\ \frac{1}{k}\pi_1 =& \frac{1}{2k}\pi_1 + \frac{1}{2}\frac{1}{(k+1)k}\pi_1 + \frac{1}{2}\pi_{k+1} \\ \implies \pi_{k+1} =& \left(\frac{2}{k} - \frac{1}{k} - \frac{1}{(k+1)k}\right)\pi_1 \\ \pi_{k+1} =& \frac{1}{k+1}\pi_1 \end{aligned}$$

However, this shows that no such stationary distribution exists, since if we attempt to normalise this relation we find

$$\sum_{i=1}^{\infty} \pi_i = \sum_{i=1}^{\infty} \frac{1}{i}\pi_1 = \pi_1 \sum_{i=1}^{\infty} \frac{1}{i}$$

Which is a divergent series and hence the stationary distribution can never be normalised. This then implies that this Markov chain has no stationary distribution.

3g.

Question 1.5.4, pg 29 [MC-1].

1.5.4 A random sequence of non-negative integers $(F_n)_{n\geq 0}$ is obtained by setting $F_0 = 0$ and $F_1 = 1$ and, once $F_0, \dots, F_n$ are known, taking F_{n+1} to be either the sum or the difference of F_{n-1} and F_n, each with probability 1/2. Is $(F_n)_{n\geq 0}$ a Markov chain?

By considering the Markov chain $X_n = (F_{n-1}, F_n)$, find the probability that $(F_n)_{n\geq 0}$ reaches 3 before first returning to 0.

Draw enough of the flow diagram for $(X_n)_{n\geq 0}$ to establish a general pattern. Hence, using the strong Markov property, show that the hitting probability for $(1, 1)$, starting from $(1, 2)$, is $(3 - \sqrt{5})/2$.

Deduce that $(X_n)_{n\geq 0}$ is transient. Show that, moreover, with probability 1, $F_n \to \infty$ as $n \to \infty$.

· $(F_n)_{n\geq 0}$ is not a Markov Chain as it fails the Markov Property: the next state in the sequence is not only dependent on the previous state

i.e. $\mathbb{P}(F_{n+1}=j \mid F_n=i, F_{n-1}=i_{n-1}, \dots, F_0=0) \neq \mathbb{P}(F_{n+1}=j \mid F_n=i) \quad \forall i, j, n$

If we now consider $X_n = (F_{n-1}, F_n)$. We see that if we let $h(x, y)$ be the probability of F_n hitting 3 before hitting 0 given we start in $X_0 = (0, 1)$. Then $h(x, 0) = 0$ and $h(x, 3) = 1$, which gives

$$h(1,1) = \frac{1}{2}h(1,0) + \frac{1}{2}h(1,2)$$
$$h(1,2) = \frac{1}{2}h(2,1) + \frac{1}{2}h(2,3)$$
$$h(2,1) = \frac{1}{2}h(1,1) + \frac{1}{2}h(1,3)$$

Substituting in $h(x, 0) = 0$ and $h(x, 3) = 1$ we get

$$h(1,1) = \frac{1}{2}h(1,2)$$
$$h(1,2) = \frac{1}{2}h(2,1) + \frac{1}{2}$$
$$h(2,1) = \frac{1}{2}h(1,1) + \frac{1}{2}$$

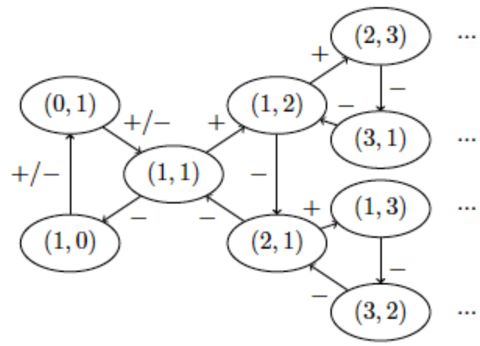
Solving this we find,

$$\begin{aligned} h(1,1) &= \frac{1}{2}h(1,2) \\ &= \frac{1}{2}\left(\frac{1}{2}h(2,1) + \frac{1}{2}\right) = \frac{1}{4}h(2,1) + \frac{1}{4} \\ &= \frac{1}{4}\left(\frac{1}{2}h(1,1) + \frac{1}{2}\right) + \frac{1}{4} = \frac{1}{8}h(1,1) + \frac{3}{8} \\ \implies h(1,1) &= \frac{3}{7} \end{aligned}$$

Since $\mathbb{P}(X_1 = (1,1) \mid X_0 = (0,1)) = 1$ we can conclude that the probability of F_n reaching 3 before returning to 0 is $\frac{3}{7}$.

Or

[redacted]. The flow diagram for $(X_n)_{n \geq 0}$ appears as follows:



One can observe that there are two ways to get to $F_i = 3$ for some i. We can follow the path on the top half of the tree $(0,1) \to (1,1) \to (1,2) \to (2,3)$, or the path $(0,1) \to (1,1) \to (1,2) \to (2,1) \to (1,3)$. The first step has probability 1 since taking either an adding or subtracting step will take you to $(1,1)$ regardless (as can be seen on the tree diagram). The second, third and fourth steps have probability $1/2$ as they require a specific operation to be chosen.

We can also note from observing the tree that when we reach $(1,2)$, we can also make the transition $(1,2) \to (2,1) \to (1,1) \to (1,2)$, which has probability $\frac{1}{8}$ from $(1,2)$. This just takes us back to where we started, and we can do this as many times as we want. So for the hitting time calculation, we need to take into account an additional $\frac{1}{8^c}$, where c is the number of cycles we make going back to the same state. We can encode this by simply computing the sum

$$\sum_{i=0}^{\infty} \frac{1}{8^i} = \frac{a}{1-r} = \frac{1}{\frac{7}{8}} = \frac{8}{7}.$$

Therefore our probability that we reach 3 before getting to 0 is given by

$$1 \times \frac{1}{2} \times \frac{1}{2} \times \frac{8}{7} + 1 \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{8}{7} = \frac{2}{7} + \frac{1}{7} = \boxed{\frac{3}{7}}.$$

To Do

3h.

Question 1.8.4, pg 46 [MC-1].

1.8.4 Each morning a student takes one of the three books he owns from his shelf. The probability that he chooses book i is α_i, where $0 < \alpha_i < 1$ for $i = 1, 2, 3$, and choices on successive days are independent. In the evening he replaces the book at the left-hand end of the shelf. If p_n denotes the probability that on day n the student finds the books in the order 1,2,3, from left to right, show that, irrespective of the initial arrangement of the books, p_n converges as $n \to \infty$, and determine the limit.

This problem can be thought of as a Markov Chain with state space =

$\{123, 132, 213, 231, 312, 321\}$,

representing the order of the books.

The transition probability matrix for this MC can be seen below:

$$P = \begin{array}{c} \\ 123 \\ 132 \\ 213 \\ 231 \\ 312 \\ 321 \end{array} \begin{array}{c} \begin{array}{cccccc} 123 & 132 & 213 & 231 & 312 & 321 \end{array} \\ \begin{bmatrix} \alpha_1 & 0 & \alpha_2 & 0 & \alpha_3 & 0 \\ 0 & \alpha_1 & \alpha_2 & 0 & \alpha_3 & 0 \\ \alpha_1 & 0 & \alpha_2 & 0 & 0 & \alpha_3 \\ \alpha_1 & 0 & 0 & \alpha_2 & 0 & \alpha_3 \\ 0 & \alpha_1 & 0 & \alpha_2 & \alpha_3 & 0 \\ 0 & \alpha_1 & 0 & \alpha_2 & 0 & \alpha_3 \end{bmatrix} \end{array}$$

If we can find $\pi = [\pi_{123} \ \pi_{132} \ \pi_{213} \ \pi_{231} \ \pi_{312} \ \pi_{321}]$

such that $\text{Sum}(\pi) = 1$ and $\pi = \pi P$, then π is the

stationary distribution and p_n converveges as $n \to \infty$ to π_{123}

$$
\begin{aligned}
\pi_{123} =& \alpha_1\Big(\pi_{123} + \pi_{213} + \pi_{231}\Big) \\
\pi_{132} =& \alpha_1\Big(\pi_{132} + \pi_{312} + \pi_{321}\Big) \\
\pi_{213} =& \alpha_2\Big(\pi_{123} + \pi_{132} + \pi_{213}\Big) \\
\pi_{231} =& \alpha_2\Big(\pi_{123} + \pi_{231} + \pi_{321}\Big) \\
\pi_{312} =& \alpha_3\Big(\pi_{123} + \pi_{132} + \pi_{312}\Big) \\
\pi_{123} + \pi_{132} + \pi_{213} + \pi_{231} + \pi_{312} + \pi_{321} &= 1
\end{aligned}
$$

From this we see that

$$
\begin{aligned}
\pi_{123} =& \frac{\alpha_1(\pi_{213} + \pi_{231})}{1 - \alpha_1} \\
\pi_{132} =& \frac{\alpha_1(\pi_{312} + \pi_{321})}{1 - \alpha_1} \\
\pi_{213} =& \frac{\alpha_2(\pi_{123} + \pi_{132})}{1 - \alpha_2} \\
\pi_{231} =& \frac{\alpha_2(\pi_{123} + \pi_{321})}{1 - \alpha_2} \\
\pi_{312} =& \frac{\alpha_3(\pi_{123} + \pi_{132})}{1 - \alpha_3} \\
\pi_{321} =& 1 - \sum_{ijk \neq 312} \pi_{ijk}
\end{aligned}
$$

Back substituting we find a stationary distribution of

$$
\begin{aligned}
\pi_{123} =& \frac{\alpha_1\alpha_2}{1 - \alpha_1} \\
\pi_{132} =& \frac{\alpha_1\alpha_2 - \alpha_1 + \alpha_1^2}{\alpha_1 - 1} \\
\pi_{213} =& \frac{\alpha_1\alpha_2}{1 - \alpha_2} \\
\pi_{231} =& \frac{\alpha_1\alpha_2 - \alpha_2 + \alpha_2^2}{\alpha_2 - 1} \\
\pi_{312} =& \frac{\alpha_1\alpha_3}{1 - \alpha_3} \\
\pi_{321} =& \frac{\alpha_2\alpha_2}{1 - \alpha_3}
\end{aligned}
$$

Therefore the limit of p_n as $n \to \infty$ is $\pi_{123} = \frac{\alpha_1\alpha_2}{1-\alpha_1}$.

3i.

Consider Theorem 1.8.3 on pg 41 [MC-1]. Rewrite the theorem and the proof specializing to two state Markov chains on the state space $\{1, 2\}$.

Theorem 1.8.3 (Convergence to equilibrium). *Let P be irreducible and aperiodic, and suppose that P has an invariant distribution π. Let λ be any distribution. Suppose that $(X_n)_{n\geq 0}$ is Markov(λ, P). Then*

$$\mathbb{P}(X_n = j) \to \pi_j \quad \text{as } n \to \infty \text{ for all } j.$$

In particular,

$$p_{ij}^{(n)} \to \pi_j \quad \text{as } n \to \infty \text{ for all } i, j.$$

To understand this proof one should see what goes wrong when P is not aperiodic. Consider the two-state chain of Example 1.8.1 which has $(1/2, 1/2)$ as its unique invariant distribution. We start $(X_n)_{n\geq 0}$ from 0 and $(Y_n)_{n\geq 0}$ with equal probability from 0 or 1. However, if $Y_0 = 1$, then, because of periodicity, $(X_n)_{n\geq 0}$ and $(Y_n)_{n\geq 0}$ will never meet, and the proof fails. We move on now to the cases that were excluded in the last theorem, where $(X_n)_{n\geq 0}$ is periodic or transient or null recurrent. The remainder of this section might be omitted on a first reading.

See general coupling proof.

4a.

Derive the chain binomial representation in Equation (1.4.2) of pg 12 [EM-1].

$$
\begin{aligned}
L &= P\{X_1, Y_1 \mid X_0, Y_0\}\, P\{X_2, Y_2 \mid X_1, Y_1\}\, P\{X_2, 0 \mid X_2, Y_2\} \\
&= \binom{X_0}{X_1}(1-q^{Y_0})^{Y_1} q^{Y_0 X_1} \binom{X_1}{X_2}(1-q^{Y_1})^{Y_2} q^{Y_1 X_2} \binom{X_2}{X_2}(1-q^{Y_2})^0 q^{Y_2 X_2} \\
&= \binom{X_0}{X_1}\binom{X_1}{X_2}(1-q^{Y_0})^{Y_1}(1-q^{Y_1})^{Y_2} q^{Y_0 X_1 + Y_1 X_2 + Y_2 X_2}, \qquad (1.4.2)
\end{aligned}
$$

In the Reed-Frost model the conditional probability of a state $(X,Y)_{t+1}$, where X_{t+1} and Y_{t+1} are the number of susceptible and infectives at time $t+1$, given $(X,Y)_t$ is

$$
\mathbb{P}((X,Y)_{t+1}|(X,Y)_t) = \binom{X_t}{X_{t+1}}(1-q^{Y_t})^{Y_{t+1}}\left(q^{Y_t}\right)^{X_{t+1}}
$$

Where q is the probability of a single susceptible not making adequate contact with a single infective to contract the epidemic.

The probability of an epidemic L to produce the sample paths shown in Figure 1.3 [EM-1] is, where $(x,y)_3 = (x_2, 0)$ and $(X,Y)_0 = (x,y)_0$ is

$$
\begin{aligned}
L &= \mathbb{P}((X,Y)_3, (X,Y)_2, (X,Y)_1, (X,Y)_0) \\
&= \mathbb{P}((X_2,0)|(X,Y)_2)\mathbb{P}((X,Y)_2|(X,Y)_1)\mathbb{P}((X,Y)_1|(X,Y)_0)\mathbb{P}((X,Y)_0 = (x,y)_0) \\
&= \binom{X_2}{X_2}(1-q^{Y_2})^0\left(q^{Y_2}\right)^{X_2} \times \binom{X_1}{X_2}(1-q^{Y_1})^{Y_2}\left(q^{Y_1}\right)^{X_2} \times \binom{X_0}{X_1}(1-q^{Y_0})^{Y_0}\left(q^{Y_0}\right)^{X_1} \times 1 \\
&= \left(q^{Y_2}\right)^{X_2}\binom{X_1}{X_2}(1-q^{Y_1})^{Y_2}\left(q^{Y_1}\right)^{X_2}\binom{X_0}{X_1}(1-q^{Y_0})^{Y_0}\left(q^{Y_0}\right)^{X_1} \\
&= \binom{X_1}{X_2}\binom{X_0}{X_1} q^{Y_2 X_2}(1-q^{Y_1})^{Y_2} q^{Y_1 X_2}(1-q^{Y_0})^{Y_0} q^{Y_0 X_1} \\
&= \binom{X_1}{X_2}\binom{X_0}{X_1} q^{Y_2 X_2 + Y_1 X_2 + Y_0 X_1}(1-q^{Y_1})^{Y_2}(1-q^{Y_0})^{Y_0}
\end{aligned}
$$

Which is the desired result.

4b.

Derive Equations (4.1.4) pg 107 [EM-4]. This was part of Project 1 as well.

$$\begin{aligned} \mathrm{E}[X_t \mid X_0 = x_0] &= \alpha^t x_0, \\ \mathrm{E}[Y_t \mid X_0 = x_0] &= \alpha^{t-1}(1-\alpha)x_0. \end{aligned} \tag{4.1.4}$$

First, we note the following one-step expectations:

$$\mathbb{E}[X_t \mid X_{t-1}] = \alpha X_{t-1}$$
$$\mathbb{E}[Y_t \mid X_{t-1}] = (1-\alpha)X_{t-1}$$

These arise from the fact that α is the rate of non-infection from a single exposure, regardless of the number of people currently infected. Now, we will use $\mathbb{E}_{x_0}$ to denote the expectation given a particular starting condition x_0 and we find:

$$\begin{aligned} \mathbb{E}_{x_0} X_t &= \mathbb{E}_{x_0}\left(\mathbb{E}[X_t \mid X_{t-1}]\right) \\ &= \mathbb{E}_{x_0}[\alpha X_{t-1}] \\ &= \alpha \mathbb{E}_{x_0} X_{t-1} \end{aligned}$$

Iterating on this gives us:

$$\mathbb{E}_{x_0} X_t = \alpha^t x_0$$

as required.

Now consider:

$$\begin{aligned} \mathbb{E}_{x_0} Y_t &= \mathbb{E}_{x_0}\left(\mathbb{E}[Y_t \mid X_{t-1}]\right) \\ &= \mathbb{E}_{x_0}[(1-\alpha)X_{t-1}] \\ &= (1-\alpha)\mathbb{E}_{x_0} X_{t-1} \end{aligned}$$

Using the expectation for X_{t-1} as calculated above, this yields:

$$\mathbb{E}_{x_0} Y_t = (1-\alpha)\alpha^{t-1} x_0$$

as required.

5a.
Question 2.23, pg 119 [SP-2].

2.23. *When did the chicken cross the road?* Suppose that traffic on a road follows a Poisson process with rate λ cars per minute. A chicken needs a gap of length at least c minutes in the traffic to cross the road. To compute the time the chicken will have to wait to cross the road, let $t_1, t_2, t_3, \ldots$ be the interarrival times for the cars and let $J = \min\{j : t_j > c\}$. If $T_n = t_1 + \cdots + t_n$, then the chicken will start to cross the road at time T_{J-1} and complete his journey at time $T_{J-1} + c$. Use the previous exercise to show $E(T_{J-1} + c) = (e^{\lambda c} - 1)/\lambda$.

From the previous exercise we know that for an exponentially distributed random variable T and some constant $c > 0$ that we have the relation

$$
\begin{aligned}
\mathbb{E}[T|T<c] =& \frac{\int_0^c tf(t)\,dt}{F(c)} = \frac{\int_0^c \lambda t e^{-\lambda t}\,dt}{1-e^{-c\lambda}} \\
=& \frac{te^{-\lambda t}\Big|_c^0 + \int_0^c e^{-\lambda t}\,dt}{1-e^{-c\lambda t}} \\
=& \frac{-ce^{-\lambda c} + \frac{1}{\lambda}\left(e^{-\lambda t}\right)\Big|_c^0}{1-e^{-c\lambda t}} \\
=& \frac{-ce^{-\lambda c} + \frac{1-e^{-\lambda c}}{\lambda}}{1-e^{-c\lambda t}} \\
=& \frac{1}{\lambda} - \frac{c}{e^{\lambda c}-1}
\end{aligned}
$$

Proof. We can view $T_{J-1} + c$ as the splitting of two different cases: one case where the first interarrival time is more than c (in which case we can immediately cross) and the other case where it is less, and we have to wait until an arrival time which is appropriate to cross. This gives us

$$T_{J-1} + c = \mathbb{1}_{\{t_1 > c\}} c + \mathbb{1}_{\{t_1 \le c\}} \left(t_1 + c + \sum_{i=2}^{J-1} t_i \right)$$

Taking the expectation gives us:

$$
\begin{aligned}
\mathbb{E}[T_{J-1} + c] &= \mathbb{P}(t_1 > c)c + \mathbb{P}(t_1 \le c)\left(\mathbb{E}[t_1 \mid t_1 \le c] + \mathbb{E}\left[\sum_{i=2}^{J-1} t_i + c \mid t_1 \le c\right]\right) \\
&= \mathbb{P}(t_1 > c)c + \mathbb{P}(t_1 \le c)\mathbb{E}[t_1 \mid t_1 \le c] + \mathbb{P}(t_1 \le c)\mathbb{E}\left[\sum_{i=2}^{J-1} t_i + c \mid t_1 \le c\right] \\
&\overset{\text{prev. ex.}}{=} ce^{-\lambda c} + -ce^{-\lambda c} + \frac{1}{\lambda}(1 - e^{-\lambda c}) + (1 - e^{-\lambda c})\mathbb{E}\left[\sum_{i=2}^{J-1} t_i + c \mid t_1 \le c\right]
\end{aligned}
$$

We can replace $\mathbb{E}[\sum_{i=2}^{J-1} t_i + c \mid t_1 \leq c]$ with $\mathbb{E}[\sum_{i=1}^{J-1} t_i + c] = \mathbb{E}[T_{J-1} + c]$ by memoryless property. Therefore, we have that

$$\begin{aligned}\mathbb{E}[T_{J-1}+c] = \frac{1}{\lambda}(1-e^{-\lambda c}) + (1-e^{-\lambda c})\mathbb{E}[T_{J-1}+c] &\implies (1-1+e^{-\lambda c})\mathbb{E}[T_{J-1}+c] = \frac{1}{\lambda}(1-e^{-\lambda c}) \\ &\implies e^{-\lambda c}\mathbb{E}[T_{J-1}+c] = \frac{1}{\lambda}(1-e^{-\lambda c}) \\ &\implies \mathbb{E}[T_{J-1}+c] = \frac{\frac{1}{\lambda}(1-e^{-\lambda c})}{e^{-\lambda c}} = \frac{1}{\lambda}(e^{-\lambda c}-1).\end{aligned}$$

OR

$$\begin{aligned}\text{Recall, } \mathbb{E}[T_{J-1} \mid J] &= \mathbb{E}[\sum_{i=1}^{J-1} t_i \mid J] = \sum_{i=1}^{J-1} \mathbb{E}[t_i \mid t_i \leq c] \\ &= \sum_{i=1}^{J-1} \frac{e^{\lambda c} - \lambda c - 1}{\lambda(e^{\lambda c - 1})} \\ &= \frac{e^{\lambda c} - \lambda c - 1}{\lambda(e^{\lambda c - 1})} \cdot (J-1) \\ \text{Now, } \mathbb{E}[T_{J-1} + c] &= \mathbb{E}\mathbb{E}[T_{J-1} \mid J] + c \\ &= \frac{e^{\lambda c} - \lambda c - 1}{\lambda(e^{\lambda c - 1})}\mathbb{E}[J-1] + c\end{aligned}$$

Again recall the definition of $J = \min\{j \mid t_j > c\}$. J is the first inter-event time with duration longer than c. Since t_i are iid, they all have the same underlying chance that they are greater than c, which is given by the CCDF; $\mathbb{P}(t_i > c) = e^{-\lambda c}$. But we are repeatedly asking this question of every t_i. So the probability of "success", that any of these will be greater than c is geometrically distributed, with rate given by the underlying distribution. For $X \sim \mathsf{geom(p)}, \mathbb{E}X = \frac{1}{p}$

$$\begin{aligned}\implies \mathbb{E}J &= \frac{1}{e^{-\lambda c}} \\ \implies \mathbb{E}[J-1] &= e^{\lambda c} - 1 \\ \implies \mathbb{E}[T_{J-1}+c] &= \frac{e^{\lambda c} - \lambda c - 1}{\lambda(e^{\lambda c - 1})}(e^{\lambda c} - 1) + c \\ &= \frac{e^{\lambda c} - \lambda c - 1}{\lambda} + c = \frac{e^{\lambda c} - 1}{\lambda}\end{aligned}$$

5b.

Question 2.30, pg 120 [SP-2].

2.30. Suppose that the number of calls per hour to an answering service follows a Poisson process with rate 4. Suppose that 3/4's of the calls are made by men, 1/4 by women, and the sex of the caller is independent of the time of the call. (a) What is the probability that in one hour exactly two men and three women will call the answering service? (b) What is the probability 3 men will make phone calls before three women do?

Let the number of calls in the t-th hour be X_t, the total number of calls till the t-th hour be S_t. The number of calls per hour follows Poisson process with rate 4, so X_t has independent poisson distribution with rate 4, and S_t the total number of calls in time t, has a Poisson distribution with mean 4t.

(a)

The total number of calls in an hour has Poisson distribution with mean 4, by the thinning theorem, the calls from men and women are independent Poisson process with rate 3 and 1 respectively. Thus the probability of interest is:

$$P(Women = 3, Men = 2) = P(Women = 3)P(Men = 2) = e^{-1}1^3/3! * e^{-3}3^2/2! = 0.0137$$

Thus, the probability that in one hour exactly two men and three women will call the answering service is 0.0137.

(b)

Note that the event in the question is equivalent to having at least 3 men calling in the first 5 phone calls, and we know the probability of the call from a man is 3/4. So the probability of interest is:

$$\begin{aligned} P(\text{at least 3 men calling in 5 calls}) &= \sum_{k=3}^{5} \binom{5}{k} 0.75^k 0.25^{(5-k)} \\ &= \binom{5}{3}0.75^3 0.25^{(5-3)} + \binom{5}{4}0.75^4 0.25^{(5-4)} + \binom{5}{5}0.75^5 0.25^{(5-5)} \\ &= 0.2637 + 0.3955 + 0.2373 \\ &= 0.8965 \end{aligned}$$

Thus, the probability that 3 men will call before three women is 0.8965.

5c.

Question 2.43, pg 122 [SP-2].

2.43. A policewoman on the evening shift writes a Poisson mean 6 number of tickets per hour. 2/3's of these are for speeding and cost \$100. 1/3's of these are for DWI and cost \$400. (a) Find the mean and standard deviation for the total revenue from the tickets she writes in an hour. (b) What is the probability that between 2 a.m. and 3 a.m. she writes five tickets for speeding and one for DWI. (c) Let A be the event that she writes no tickets between 1 a.m. and 1:30, and N be the number of tickets she writes between 1 a.m. and 2 a.m. Which is larger $P(A)$ or $P(A|N = 5)$? Don't just answer yes or no, compute both probabilities.

Assuming that tickets for speeding are written independently of those for DWI we can model the number of speeding tickets written as $\mathsf{Pois}(\frac{2}{3} \times 6 = 4)$ and the number of DWI tickets as $\mathsf{Pois}(\frac{1}{3} \times 6 = 2)$. Therefore the expected revenue per hour is

$$\$100 \times 4 + \$400 \times 2 = \$1200$$

and the standard deviation is

$$\sqrt{100^2 \times 4 + 400^2 \times 2} = 600$$

Since these two Poisson Processes are independent

$$\begin{aligned}\mathbb{P}(S = 5, DWI = 1) =& \mathbb{P}(S = 5)\mathbb{P}(DWI = 1) \\ =& \frac{1}{5!}4^5 e^{-4} \frac{1}{1!} 2^1 e^{-2} \\ =& \frac{256}{15} e^{-6} \approx 0.0423\end{aligned}$$

To find $\mathbb{P}(A)$ we once again use the independence. However, since this is only a half hour period we must half the rates. So now $S \sim \mathsf{Pois}(2)$ and $DWI \sim \mathsf{Pois}(1)$

$$\begin{aligned}\mathbb{P}(A) &= \mathbb{P}(S = 0)\mathbb{P}(DWI = 0) \\ &= e^{-2}e^{-1} \\ &\approx 0.0498\end{aligned}$$

The number of tickets written in an hour N follows a $\mathsf{Pois}(6)$ distribution. Therefore

$$\begin{aligned}\mathbb{P}(A|N = 5) =& \frac{\mathbb{P}(A \cap N = 5)}{\mathbb{P}(N = 5)} = \frac{\mathbb{P}(\mathsf{Pois}(3) = 5)}{\mathbb{P}(\mathsf{Pois}(6) = 5)} \quad \text{By the memoryless property} \\ =& \frac{\frac{1}{5!}3^5 e^{-5}}{\frac{1}{5!}6^5 e^{-5}} \\ =& \frac{1}{2^5} \approx 0.0313\end{aligned}$$

So $\mathbb{P}(A)$ is larger than $\mathbb{P}(A|N = 5)$

5d.
Question 2.47, pg 123 [SP-2].

2.47. Let S and T be exponentially distributed with rates λ and μ. Let $U = \min\{S,T\}$ and $V = \max\{S,T\}$. Find (a) EU. (b) $E(V-U)$, (c) EV. (d) Use the identity $V = S+T-U$ to get a different looking formula for EV and verify the two are equal.

a) $\mathbb{E}(U)$

First find the CDF of U. Since $s\in S$ and $t\in T$ are always positive, $u\in U$ is always positive. Therefore, for some $u>0$,

$$\mathbb{P}(U>) = \mathbb{P}(\min\{S,T\}>u)$$
$$= \mathbb{P}(S>u,\ T>u)$$
$$= \mathbb{P}(S>u)\ \mathbb{P}(T>u)$$
$$= e^{-u\lambda}e^{-u\mu}$$
$$= e^{-u(\lambda+\mu)}$$

Therefore $1-\mathbb{P}(U>u) = 1-e^{-u(\lambda+\mu)}$

Note that this is the form of the CDF of an exponential distribution, so U is exponentially distributed with rate $\lambda+\mu$.

$$\therefore\ \mathbb{E}(U) = \frac{1}{\lambda+\mu}$$

b) $\mathbb{E}(V-U)$

Note that $V=\max\{S,T\}$ and $U=\min\{S,T\}$, so $V=S+T-U$

Therefore,

$$\mathbb{E}(V-U) = \mathbb{E}(S+T-2U)$$
$$= \mathbb{E}(S)+\mathbb{E}(T)-2\mathbb{E}(U)$$
$$= \frac{1}{\lambda}+\frac{1}{\mu}-\frac{2}{\lambda+\mu}$$

c) $\mathbb{E}(V)$

$V=\max\{X,Y\}$

$$\mathbb{P}(V\le v) = \mathbb{P}(S\le v,\ T\le v) = \mathbb{P}(S\le v)\ \mathbb{P}(T\le v)$$
$$= \left(1-\mathbb{P}(S>v)\right)\left(1-\mathbb{P}(T>v)\right)$$
$$= \left(1-e^{-v\lambda}\right)\left(1-e^{-v\mu}\right)$$
$$= 1-e^{-v\lambda}-e^{-v\mu}+e^{v(\lambda+\mu)}$$

Using $\mathbb{E}(V) = \int_0^\infty 1-\mathbb{P}(V\le v)\,dv$

We can write

$$\mathbb{E}(V) = \int_0^\infty e^{-v\lambda} + e^{-v\mu} - e^{-v(\lambda+\mu)}\, dv$$

$$= \left[\frac{-e^{-v\lambda}}{\lambda} - \frac{e^{-v\mu}}{\mu} + \frac{e^{-v(\lambda+\mu)}}{(\lambda+\mu)} \right]_0^\infty$$

$$= -0 - 0 + 0 - \left(-\frac{1}{\lambda} - \frac{1}{\mu} + \frac{1}{(\lambda+\mu)} \right)$$

$$\mathbb{E}(V) = \frac{1}{\lambda} + \frac{1}{\mu} - \frac{1}{\lambda+\mu}$$

d) Use the identity $V = S + T - U$ to get a different looking formula for $\mathbb{E}V$ and verify that the two are equal.

Since $V = S + T - U$,

$$\mathbb{E}(V) = \mathbb{E}(S + T - U)$$

$$= \mathbb{E}(S) + \mathbb{E}(T) - \mathbb{E}(U)$$

$$= \frac{1}{\lambda} + \frac{1}{\mu} - \frac{1}{\lambda+\mu}$$

This matches what we found in (c).

5e.

Consider the algorithm for generating a Poisson random variable with mean λ using Equation (3.15) pg 100 [SWJ-3]. Explain (prove) why this algorithm works.

The Poisson process possesses many elegant analytic properties, and these sometimes come as an aid when considering Poisson distributed random variables. One such (seemingly magical) property is to consider the random variable $N \geq 0$ such that,

$$\prod_{i=1}^{N} U_i \geq e^{-\lambda} > \prod_{i=1}^{N+1} U_i, \tag{3.15}$$

where $U_1, U_2, \ldots$ is a sequence of i.i.d. uniform$(0,1)$ random variables and $\prod_{i=1}^{0} U_i \equiv 1$. It turns out that seeking such a random variable N produces an efficient recipe for generating a Poisson random variable. That is, the N defined by (3.15) is Poisson distributed with mean λ. Notice that the recipe dictated by (3.15) is to continue multiplying uniform random variables to a "running product" until the product goes below the desired level $e^{-\lambda}$.

In a Poisson process, each arrival are iid exponentially distributed, with parameter λ. Then to find how many arrives by time $t = 1$, we take the random variable K to the last time when its cumulative sum is still less than 1. i.e. we look at

$$\sum_{i=1}^{K} E_i \leq 1 < \sum_{i=1}^{K+1} E_i$$

where each $E_i \sim \mathsf{Exp}(\lambda)$ iid. So K as a random variable is distributed according to $N(1) \sim \mathsf{Poi}(\lambda)$, by the reasoning above.

$1 - U_i \overset{d}{=} U_i$

By inverse transform, we have $E_i = -\ln(U_i)/\lambda$, so the above equation becomes

$$\sum_{i=1}^{K} -\ln(U_i)/\lambda \leq 1 < \sum_{i=1}^{K+1} -\ln(U_i)/\lambda$$

$$\sum_{i=1}^{K} \ln(U_i) \geq -\lambda > \sum_{i=1}^{K+1} \ln(U_i)$$

$$\prod_{i=1}^{K} U_i \geq e^{-\lambda} > \prod_{i=1}^{K+1} U_i$$

as required.

5f.

Consider Theorem 2.4.6 pg 79 [MC-2]. Rewrite the theorem and the proof, specializing to the case of n = 2. Plot/sketch the joint distribution of J1 and J2 in this case, with X10 = 2.

Theorem. *Let $(X_t)_{t\geq 0}$ be a Poisson process. Then, condition on the event $\{X_t = 2\}$, the jump times J_1, J_2 have joint density function*

$$f(t_1, t_2) = \frac{2}{t^2}\mathbb{1}_{\{0\leq t_1 \leq t_2 \leq t\}}.$$

Thus, condition on $\{X_t = 2\}$, the jump times J_1, J_2 have the same distribution as an ordered sample of size 2 from the uniform distribution on $[0, t]$.

Proof. The holding times S_1, S_2, S_3 have joint density $\lambda^3 e^{-\lambda(s_1+s_2+s_3)}\mathbb{1}_{\{s_1,s_2,s_3\geq 0\}}$, so the jump times J_1, J_2, J_3 have joint density function $\lambda^3 e^{-\lambda t_3}\mathbb{1}_{\{0\leq t_1\leq t_2\leq t_3\}}$. So, for $A \subseteq \mathbb{R}^n$ we have

$$\begin{aligned}\mathbb{P}((J_1, J_2) \in A \cap X_t = 2) &= \mathbb{P}((J_1, J_2) \in A \cap J_2 \leq t \leq J_3)\\ &= e^{-\lambda t}\lambda^2 \int_{t_1\in A}\int_{t_2\in A} \mathbb{1}_{\{0\leq t_1\leq t_2\leq t\}}\, dt_1\, dt_2\,.\end{aligned}$$

Note that $\mathbb{P}(X_t = 2) = e^{-\lambda t}(\lambda t)^2/2! = (e^{-\lambda t}\lambda^2 t^2)/2$ we obtain by the conditional probability formula,

$$\mathbb{P}((J_1, J_2) \in A \mid X_t = 2) = \frac{\mathbb{P}((J_1, J_2) \in A \cap X_t = 2)}{\mathbb{P}(X_t = 2)},$$

that

$$\begin{aligned}\mathbb{P}((J_1, J_2) \in A \mid X_t = 2) &= \frac{e^{-\lambda t}\lambda^2 \int_{t_1\in A}\int_{t_2\in A} \mathbb{1}_{\{0\leq t_1\leq t_2\leq t\}}\, dt_1\, dt_2\,.}{(e^{-\lambda t}\lambda^2 t^2)/2}\\ &= \int_{t_1\in A}\int_{t_2\in A} \frac{2}{t^2}\mathbb{1}_{\{0\leq t_1\leq t_2\leq t\}}\, dt_1\, dt_2 = \int_{t_1\in A}\int_{t_2\in A} f(t_1, t_2)\, dt_1\, dt_2\end{aligned}$$

with joint density function $f(t_1, t_2) = \frac{2}{t^2}\mathbb{1}_{\{0\leq t_1\leq t_2\leq t\}}$ as required. ■

which establishes the distribution function as required. This can be sketched as a 3d function in the following way:

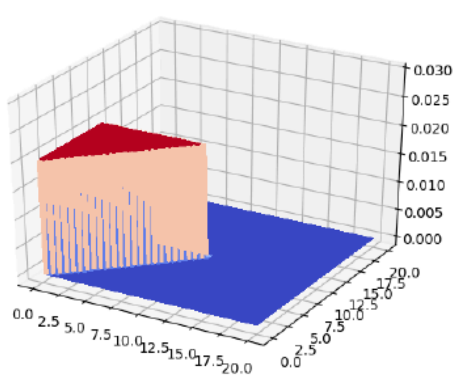


Figure One: Sketch of the distribution function

6a.
Question 4.7, pg 193 [SP-4].

4.7. Two people are working in a small office selling shares in a mutual fund. Each is either on the phone or not. Suppose that calls come in to the two brokers at rate $\lambda_1 = \lambda_2 = 1$ per hour, while the calls are serviced at rate $\mu_1 = \mu_2 = 3$. (a) Formulate a Markov chain model for this system with state space $\{0, 1, 2, 12\}$ where the state indicates who is on the phone. (b) Find the stationary distribution. (c) Suppose they upgrade their telephone system so that a call to one line that is busy is forwarded to the other phone and lost if that phone is busy. Find the new stationary probabilities. (d) Compare the rate at which calls are lost in the two systems.

(a)

Let X_t be the state at time t, where the state space is given by all possible combination of who is on the phone, $\{0, 1, 2, 12\}$ (i.e. no one is on the phone, broker 1 is on the phone, broker 2 is on the phone, both brokers are on the phone).

The generator matrix Q of the this Markov chain is given by:

	(0)	(1)	(2)	(12)
(0)	$-(\lambda_1+\lambda_2)$	λ_1	λ_2	0
(1)	μ_1	$-(\mu_1+\lambda_2)$	0	λ_2
(2)	μ_2	0	$-(\mu_2+\lambda_1)$	λ_1
(12)	0	μ_2	μ_1	$-(\mu_1+\mu_2)$

Substitute valus $\lambda_1 = \lambda_2 = 1$, $\mu_1 = \mu_2 = 3$, we get the generator matrix Q =

	(0)	(1)	(2)	(12)
(0)	−2	1	1	0
(1)	3	−4	0	1
(2)	3	0	−4	1
(12)	0	3	3	−6

(b)

For our situation, this leads us to the following four equations:

$$-2\pi_1 + 3\pi_2 + 3\pi_3 = 0$$
$$\pi_1 - 4\pi_2 + 3\pi_4 = 0$$
$$\pi_1 - 4\pi_3 + 3\pi_4 = 0$$
$$\pi_1 + \pi_2 + \pi_3 + \pi_4 = 1$$

Therefore we wish to solve

$$\begin{pmatrix}\pi_1\\ \pi_2\\ \pi_3\\ \pi_4\end{pmatrix} = \begin{pmatrix}-2 & 3 & 3 & 0\\ 1 & -4 & 0 & 3\\ 1 & 0 & -4 & 3\\ 1 & 1 & 1 & 1\end{pmatrix}^{-1}\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

Since this is quite a large system to solve by hand, we solve it using a symbolic solver to obtain the stationary distribution

$$\boxed{(\pi_1, \pi_2, \pi_3, \pi_4) = 9/16,\ 3/16,\ 3/16,\ 1/16}.$$

For our situation, this leads us to the following four equations:

$$
\begin{aligned}
-2\pi_1 + 3\pi_2 + 3\pi_3 &= 0 \\
\pi_1 - 4\pi_2 + 3\pi_4 &= 0 \\
\pi_1 - 4\pi_3 + 3\pi_4 &= 0 \\
\pi_1 + \pi_2 + \pi_3 + \pi_4 &= 1
\end{aligned}
$$

Therefore we wish to solve

$$
\begin{pmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \\ \pi_4 \end{pmatrix} = \begin{pmatrix} -2 & 3 & 3 & 0 \\ 1 & -4 & 0 & 3 \\ 1 & 0 & -4 & 3 \\ 1 & 1 & 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}
$$

Since this is quite a large system to solve by hand, we solve it using a symbolic solver to obtain the stationary distribution

$$
\boxed{(\pi_1, \pi_2, \pi_3, \pi_4) = 9/16,\ 3/16,\ 3/16,\ 1/16}.
$$

Compare the rate at which calls are lost in the two systems.

From observation we see that in new system only state 12 will lose calls, whereas in the original system states $1, 2$ and 12 will all lose calls. In the second system in over the long term the system will be in state 12 for $\frac{2}{17}$ of an hour. While it is in that state calls will be coming in to both phones at a rate $\lambda_1 + \lambda_2 = 2$ per hour. Therefore, the rate that calls are lost will be

$$
\mu_{2_{lost}} = (\lambda_1 + \lambda_2)\pi_{12} = \frac{4}{17}
$$

In the first system this rate will be

$$
\mu_{1_{lost}} = \lambda_1\pi_1 + \lambda_2\pi_2 + (\lambda_1 + \lambda_2)\pi_{12} = \frac{3}{16} + \frac{3}{16} + \frac{1}{8} = \frac{1}{2}
$$

Therefore, the old system loses calls at a rate $\mu_{1_{lost}} = \frac{1}{2}$ which is a much higher rate than the new system $\mu_{2_{lost}} = \frac{4}{17}$.

6b.
Question 4.14, pg 194 [SP-4].

4.14. A small company maintains a fleet of four cars to be driven by its workers on business trips. Requests to use cars are a Poisson process with rate 1.5 per day. A car is used for an exponentially distributed time with mean 2 days. Forgetting about weekends, we arrive at the following Markov chain for the number of cars in service:

	0	**1**	**2**	**3**	**4**
0	–1.5	1.5	0	0	0
1	0.5	–2.0	1.5	0	0
2	0	1.0	–2.5	1.5	0
3	0	0	1.5	–3	1.5
4	0	0	0	2	–2

(a) Find the stationary distribution. (b) At what rate do unfulfilled requests come in? How would this change if there were only three cars? (c) Let $g(i) = E_i T_4$. Write and solve equations to find the $g(i)$. (d) Use the stationary distribution to compute $E_3 T_4$.

Once again, the stationary distribution is the unique normalised solution to $\boldsymbol{\pi} Q = \mathbf{0}$. Solving for the stationary distribution we get

$$\begin{aligned}
\frac{3}{2}\pi_0 &= \frac{1}{2}\pi_1 \implies \pi_0 = \frac{1}{3}\pi_1 \\
2\pi_1 &= \frac{3}{2}\pi_0 + \pi_2 \implies \pi_2 = \frac{9}{2}\pi_0 \\
\frac{5}{2}\pi_2 &= \frac{3}{2}\pi_1 + \frac{3}{2}\pi_3 \implies \pi_3 = \frac{9}{2}\pi_0 \\
2\pi_4 &= \frac{3}{2}\pi_3 \implies \pi_4 = \frac{27}{8}\pi_0
\end{aligned}$$

$$\pi_0 + \pi_1 + \pi_2 + \pi_3 + \pi_4 = 1 \implies \pi_0 = \frac{8}{131}$$

Which implies that the stationary distribution is

$$\pi = \left(\frac{8}{131}, \frac{24}{131}, \frac{36}{131}, \frac{36}{131}, \frac{27}{131}\right)$$

b) Requests come in at rate 1.5 per day. These requests are unfulfilled only when the chain is in state (4). Thus the rate of unfulfilled requests is:

$$\pi_4 \times 1.5 = \frac{27}{131} \times 1.5 = \frac{81}{262} \text{ unfulfilled requests/day} \approx 0.3092 \text{ unfulfilled requests/day}$$

If there were only three cars then the generator matrix would be:

$$Q' = \begin{pmatrix} -1.5 & 1.5 & 0 & 0 \\ 0.5 & -2.0 & 1.5 & 0 \\ 0 & 1.0 & -2.5 & 1.5 \\ 0 & 0 & 1.5 & -1.5 \end{pmatrix}$$

This new chain has stationary distribution:

$$\begin{aligned} \boldsymbol{\pi}' &= \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1.5 & 1.5 & 0 & 1.0 \\ 0.5 & -2.0 & 1.5 & 1.0 \\ 0 & 1.0 & -2.5 & 1.0 \\ 0 & 0 & 1.5 & 1.0 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} \frac{1}{13} & \frac{3}{13} & \frac{9}{26} & \frac{9}{26} \end{pmatrix} \end{aligned}$$

So the new rate of unfulfilled requests would be:

$$\pi'_3 \times 1.5 = \frac{9}{26} \times 1.5 = \frac{27}{52} \text{ unfulfilled requests/day} \approx 0.5192 \text{ unfulfilled requests/day}$$

So the rate of unfulfilled requests per day would increase by about 0.21 if there were only three cars instead of four.

) Let $g(i) = \mathbb{E}_i T_4$. Write and solve equations to find $g(i)$.

The exit times $g(i) = \mathbb{E}_i T_4$ can be found via

$$\sum_i Q(j,i)g(i) = -1 \text{ for } j \notin \{4\}$$

Therefore, we get the set of equations

$$\begin{aligned}
-\frac{3}{2}g(0) + \frac{3}{2}g(1) + 1 =& 0 \quad \text{for } j = 0 \\
\frac{1}{2}g(0) - 2g(1) + \frac{3}{2}g(2) + 1 =& 0 \quad \text{for } j = 1 \\
g(1) - \frac{5}{2}g(2) + \frac{3}{2}g(3) + 1 =& 0 \quad \text{for } j = 2 \\
\frac{3}{2}g(2) - 3g(3) + 1 =& 0 \quad \text{for } j = 3
\end{aligned}$$

Going through the system of equations one by one, starting from the top, we get the relations.

$$\begin{aligned}
g(1) =& g(0) \ - \frac{2}{3} \\
g(2) =& g(0) \ - \frac{14}{9} \\
g(3) =& g(0) \ - \frac{76}{27} \\
g(0) =& \frac{128}{27}
\end{aligned}$$

Which $g(i)$ for $i = \{0, 1, 2, 3\}$ is

$$(g(0), g(1), g(2), g(3)) = \left(\frac{128}{27}, \frac{110}{27}, \frac{86}{27}, \frac{52}{27}\right)$$

d) We can calculate $\mathbb{E}_3 T_4$ from the stationary distribution by recognising that the time spent in state 4 (which is exponentially distributed with mean $\mu_4 = 1/2$) followed by the time to return to state 4 from state 3 (which has mean $\mathbb{E}_3 T_4$) is an alternating renewal process. Thus we can use the formula from SP-4 Theorem 3.4 for alternating renewal processes to get:

not required for exam

$$\begin{aligned}
\pi_4 &= \frac{\mu_4}{\mu_4 + \mathbb{E}_3 T_4} \\
\frac{27}{131} &= \frac{1/2}{1/2 + \mathbb{E}_3 T_4} \\
1 + 2\mathbb{E}_3 T_4 &= \frac{131}{27} \\
\mathbb{E}_3 T_4 &= \frac{131 - 27}{27}\frac{1}{2} \\
&= \frac{52}{27}
\end{aligned}$$

This agrees with the value for $\mathbb{E}_3 T_4$ found in part c).

6c.

Question 4.20, pg 195 [SP-4].

4.20. Consider an $M/M/4$ queue with no waiting room, for example, four prostitutes hanging out on a street corner. Customers arrive at times of a Poisson process with rate 4. Each service takes an exponentially distributed amount of time with mean 1/2. If no server is free, then the customer goes away never to come back. (a) Find the stationary distribution. (b) At what rate do customers enter the system? (c) Use $W = L/\lambda_a$ to calculate the average time a customer spends in the system.

Note that we have a service time with mean $1/2$, so this means that we have on average a rate of 2 services per hour. This gives us the Q-matrix

$$Q = \begin{pmatrix} -4 & \boxed{4} & & \\ \boxed{2} & -4-2 & -2 & \\ & 4 & -4-4 & -4 \\ & & & \ddots \end{pmatrix}$$

Recall the detailed balance condition (from page 167 of [SP-4])

$$\pi(k)q(k,j) = \pi(j)q(j,k) \quad \forall j \neq k.$$

Note that if $k = n$, we have $q(n, n-1) = 2n$ and $q(n-1, n) = 4$ for all $n \in \{0, 1, 2, 3, 4\}$. This means that we have

$$\pi(n)2n = \pi(n-1)4 \quad \Longrightarrow \pi(n) = \frac{2}{n}\pi(n-1).$$

We can iterate through recursively to obtain

$$\pi(n) = \frac{2}{n}\frac{2}{n-1}\cdots\frac{2}{1}\pi(n-1) \implies \boxed{\pi(n) = \frac{2^n}{n!}\pi(0)}.$$

To find $\pi(0)$, we require that $\sum_{i=0}^{4} \frac{2^n}{n!}\pi(0) = 1$. Therefore, we have

$$\pi(0)\left(1 + 2 + 2 + \frac{8}{6} + \frac{16}{24}\right) = 1 \implies \pi(0)\left(1 + 2 + 2 + \frac{4}{3} + \frac{2}{3}\right) = 1 \boxed{\pi(0) = \frac{1}{7}}.$$

Therefore, our stationary distribution can be obtained by multiplying $(1, 2, 2, \frac{4}{3}, \frac{2}{3})$ by $\frac{1}{7}$, which gives us

$$\boxed{\pi = \left(\frac{1}{7}, \frac{2}{7}, \frac{2}{7}, \frac{4}{21}, \frac{2}{21}\right)}$$

Question 6cb

Customers ARRIVE at rate 4 but can only enter the system if there is an unoccupied server, i.e, not in state 4. From the stationary distribution, we know that we are NOT in state 4 $\left(\frac{1}{7} + \frac{2}{7} + \frac{2}{7} + \frac{4}{21}\right)$ of the time $\left(\frac{19}{21}\right)$. Therefore the rate of customers entering the system is

$\frac{19}{21} \times 4 = \frac{76}{21} \approx 3.619047619$

c) $L = \lambda_a \bar{W} \rightarrow$ long run time cost indiv in system

↳ average arrival rate

↳ long run average # of ppl in system

$L = \sum_i i \cdot \pi_i = $ ~~$1/7 + \frac{4}{7} + \frac{12}{7} + 4$~~

$= 0 \cdot 1/7 + 1 \cdot 2/7 + 4/7 + \frac{12}{21} + \frac{8}{21}$

$= \frac{38}{21}$

$W = \frac{L}{\lambda_a} = \frac{\frac{38}{21}}{\frac{76}{21}} = \frac{1}{2}$

$\equiv$ mean of exponential service time dist

6d.
Consider the M/M/∞ queue with arrival rate λ and service rate μ. Determine the variance of the stationary distribution of the number of customers in the queue.

We claim the number of people, X in the queue at any time follows a Poisson distribution with parameter λ/μ. If this is true, then $\mathbb{E}(X) = \text{Var}(X) = \lambda/\mu$. We now prove that X follows a poisson distribution with the above parameter.

$M/M/\infty$ queue can be viewed as a birth and death chain with $\lambda_i = \lambda$ and $\mu_i = i\mu$, (where λ_i is the rate of transition from state i to state $i+1$ and μ_i is the rate of transition from i to $i-1$. since with i people in the queue, each one of the i people has an independent probability of μ leaving the queue, so the rate of leaving is $i\mu$, but each person only has a rate of λ arriving. Then the stationary distribution, which satisfies the detailed balanced equation is computed as follows

$$\pi_i \lambda_i = \pi_{i+1}\mu_{i+1} \implies \pi_{i+1} = \pi_i \frac{\lambda_i}{\mu_{i+1}} = \pi_i \frac{\lambda}{(i+1)\mu}$$

Using this recurrence relation, we have

$$\pi_i = \pi_0 \prod_{j=0}^{i-1} \frac{\lambda}{(i+1)\mu} = \pi_0 \frac{\lambda^i}{(i)!\mu^i} = \pi_0 \frac{(\lambda/\mu)^i}{(i)!}$$

Thus

$$1 = \sum_{i=0}^{\infty} \pi_i = \pi_0 \left(\frac{(\lambda/\mu)^i}{(i)!} \right) = \pi_0 e^{\lambda/\mu}$$

which implies

$$\pi_0 = e^{-\lambda/\mu}, \qquad \pi_i = e^{-\lambda/\mu} \frac{(\lambda/\mu)^i}{(i)!}$$

which is the pdf of a poisson distribution. So the number of customer in the queue follows a poisson distribution and we have the mean and variance as claimed before.

6e Question 3.6.3, pg 123 [MC-3].

3.6.3 Customers arrive at a single-server queue in a Poisson stream of rate λ. Each customer has a service requirement distributed as the sum of two independent exponential random variables of parameter μ. Service requirements are independent of one another and of the arrival process. Write down the generator matrix Q of a continuous-time Markov chain which models this, explaining what the states of the chain represent. Calculate the essentially unique invariant measure for Q, and deduce that the chain is positive recurrent if and only if $\lambda/\mu < 1/2$.

To Do

6f

Consider the two state continuous time Markov chain with birth rate λ and death rate μ. Denote the state space $\{0, 1\}$. Write the forward equations and the backward equations and use one of these to find expressions for $\mathbb{P}(X_t = 1 | X_t = 0)$.

The generator matrix for this Markov chain is:

$$Q = \begin{pmatrix} -\lambda & \lambda \\ \mu & -\mu \end{pmatrix}$$

In matrix form, Kolmogorov's forward equations are:

$$p_t' = p_t Q$$
$$\Rightarrow \begin{pmatrix} p_t'(0,0) & p_t'(0,1) \\ p_t'(1,0) & p_t'(1,1) \end{pmatrix} = \begin{pmatrix} p_t(0,0) & p_t(0,1) \\ p_t(1,0) & p_t(1,1) \end{pmatrix} \begin{pmatrix} -\lambda & \lambda \\ \mu & -\mu \end{pmatrix}$$

Where $p_t(i,j) = \mathbb{P}(X_t = j | X_0 = i)$. These expand to:

$$\begin{aligned} p_t'(0,0) &= -\lambda p_t(0,0) + \mu p_t(0,1) \\ p_t'(0,1) &= \lambda p_t(0,0) - \mu p_t(0,1) \\ p_t'(1,0) &= -\lambda p_t(1,0) + \mu p_t(1,1) \\ p_t'(1,1) &= \lambda p_t(1,0) - \mu p_t(1,1) \end{aligned}$$

In matrix form, Kolmogorov's backward equations are:

$$p_t' = Q p_t$$
$$\Rightarrow \begin{pmatrix} p_t'(0,0) & p_t'(0,1) \\ p_t'(1,0) & p_t'(1,1) \end{pmatrix} = \begin{pmatrix} -\lambda & \lambda \\ \mu & -\mu \end{pmatrix} \begin{pmatrix} p_t(0,0) & p_t(0,1) \\ p_t(1,0) & p_t(1,1) \end{pmatrix}$$

These expand to:

$$\begin{aligned} p_t'(0,0) &= -\lambda p_t(0,0) + \lambda p_t(1,0) \\ p_t'(0,1) &= -\lambda p_t(0,1) + \lambda p_t(1,1) \\ p_t'(1,0) &= \mu p_t(0,0) - \mu p_t(1,0) \\ p_t'(1,1) &= \mu p_t(0,1) - \mu p_t(1,1) \end{aligned}$$

We are interested in $\mathbb{P}(X_t = 1 | X_0 = 0) = p_t(0,1)$. Using the backward equations:

$$\begin{aligned} p_t'(0,1) &= \lambda[p_t(1,1) - p_t(0,1)] \\ p_t'(1,1) &= -\mu[p_t(1,1) - p_t(0,1)] \\ \therefore [p_t(1,1) - p_t(0,1)]' &= -(\lambda + \mu)[p_t(1,1) - p_t(0,1)] \end{aligned}$$

Since $p_0(1,1) = 1$ and $p_0(0,1) = 0$ we get that $p_0(1,1) - p_0(0,1) = 1$ and hence:

$$p_t(1,1) - p_t(0,1) = e^{-(\lambda+\mu)t}$$

Substituting this back into $p_t'(0,1) = \lambda[p_t(1,1) - p_t(0,1)]$ and integrating:

$$\begin{aligned} p_t'(0,1) &= \lambda e^{-(\lambda+\mu)t} \\ p_t(0,1) &= p_0(0,1) + \int_0^t \lambda e^{-(\lambda+\mu)t'}\, dt' \\ &= 0 + \left[-\frac{\lambda}{\lambda+\mu} e^{-(\lambda+\mu)t'} \right]_{t'=0}^{t'=t} \\ &= \frac{\lambda}{\lambda+\mu} - \frac{\lambda}{\lambda+\mu} e^{-(\lambda+\mu)t} \end{aligned}$$

Thus we have:

$$\mathbb{P}(X_t = 1 | X_0 = 0) = \frac{\lambda}{\lambda+\mu} - \frac{\lambda}{\lambda+\mu} e^{-(\lambda+\mu)t}$$

6g.

Consider the M/M/1/K queue where finite K is the total system capacity (state space is {0, 1, 2, . . . , K}). Assume λ 6= μ. Find an expression for the mean number of customers in the system in steady state.

Such a formula is given in SWJ-10 at equation (10.26):

$$L_{M/M/1/K} = \frac{\rho}{1-\rho}\left(\frac{1-(K+1)\rho^K + K\rho^{K+1}}{1-\rho^{K+1}}\right)$$

Where $\rho = \frac{\lambda}{\mu} \neq 1$.

The derivation for this is quite simple. The stationary distribution is given in SP-1 as:

$$\pi(n) = \frac{\rho^n}{c}$$

Where $c = \frac{1-\rho^{K+1}}{1-\rho}$. i.e.

$$\pi(n) = \frac{\rho^n - \rho^{n+1}}{1-\rho^{K+1}}$$

We thus get the mean number of customers in the system as:

$$\begin{aligned} L &= \sum_{n=0}^{K} n\pi(n) \\ &= \frac{1}{1-\rho^{K+1}} \sum_{n=0}^{K} n\left(\rho^n - \rho^{n+1}\right) \end{aligned}$$

From WolframAlpha:

$$\sum_{n=0}^{K} n\left(\rho^n - \rho^{n+1}\right) = \frac{\rho\left(K\rho^{K+1} - (K+1)\rho^K + 1\right)}{1-\rho}$$

Substituting this in to our expression for L:

$$\begin{aligned} L &= \frac{1}{1-\rho^{K+1}} \frac{\rho\left(K\rho^{K+1} - (K+1)\rho^K + 1\right)}{1-\rho} \\ &= \frac{\rho}{1-\rho} \frac{\left(K\rho^{K+1} - (K+1)\rho^K + 1\right)}{1-\rho^{K+1}} \\ &= \frac{\rho}{1-\rho}\left(\frac{1-(K+1)\rho^K + K\rho^{K+1}}{1-\rho^{K+1}}\right) \end{aligned}$$

Which is the same as the expression in SWJ-10.

6h.

For the M/M/1/K Try to reproduce the computations for P(W > x) as in pg 335 [SWJ-10] to obtain some expression for the CCDF of the waiting time.

$$\begin{aligned}
\mathbb{P}(\mathrm{W} > x) &= \sum_{k=1}^{\infty} \mathbb{P}(\mathrm{W} > x \mid X = k)\mathbb{P}(X = k) \\
&= \sum_{k=1}^{\infty} \int_x^{\infty} f_k(u)\, du\ (1-\rho)\rho^k \\
&= \sum_{k=1}^{\infty} \int_x^{\infty} \frac{\mu^k}{(k-1)!} u^{k-1} e^{-\mu u}\, du\ (1-\rho)\rho^k \\
&= (1-\rho)\lambda \int_x^{\infty} e^{-\mu u} \sum_{k=0}^{\infty} \frac{(\lambda u)^k}{k!}\, du \\
&= (1-\rho)\lambda \int_x^{\infty} e^{-(\mu-\lambda)u}\, du \\
&= (1-\rho)\frac{\lambda}{\mu-\lambda} e^{-(\mu-\lambda)x}
\end{aligned}$$

For the M/M/1/K queue, recall from the previous expression we had that

$$\pi_n = \frac{\rho^n}{\sum_{i=0}^{k} \rho^i},$$

which we will leave in this form to avoid the split cases for now.

$$\begin{aligned}
\mathbb{P}(W > t) &= \sum_{m=1}^{K} \mathbb{P}(W > t | X = m)\mathbb{P}(X = m) \\
&= \sum_{m=1}^{K} \int_t^{\infty} f_m(s)\, ds \cdot \frac{\rho^m}{\sum_{i=1}^{m} \rho^i}
\end{aligned}$$

where we determine $f_m(s)$ from the fact that if $X = m$, the service time will be the sum of m i.i.d. exponential random variables. This has distribution $\mathsf{Gamma}(n, \mu)$, where μ recall is the service rate (and $\rho = \frac{\lambda}{\mu}$).

$$\begin{aligned}
&= \sum_{m=1}^{K} \int_t^{\infty} \frac{\mu^m}{(m-1)!} s^{m-1} e^{-s\mu}\, ds \frac{\rho^m}{\sum_{i=1}^{m} \rho^i} \\
&= \sum_{m=1}^{K} \frac{\mu^m \rho^m}{(m-1)! \sum_{i=1}^{m} \rho^i} \int_t^{\infty} s^{m-1} e^{-s\mu}\, ds
\end{aligned}$$

Le t$x = s\mu, \quad dx = \mu ds$

$$\begin{aligned}
&= \sum_{m=1}^{K} \frac{\mu^m \rho^m}{(m-1)! \sum_{i=1}^{m} \rho^i} \int_t^{\infty} \left(\frac{x}{\mu}\right)^{m-1} e^{-s\mu} \frac{dx}{\mu} \\
&= \sum_{m=1}^{K} \frac{\mu^m \rho^m}{(m-1)! \sum_{i=1}^{m} \rho^i} \frac{1}{\mu^m} \int_t^{\infty} x^{m-1} e^{-x}\, dx
\end{aligned}$$

This integral term is exactly of the form of the incomplete gamma function, $\Gamma(s, x)$ (I'm not sure this makes things any simpler)

$$\implies \mathbb{P}(W > t) = \sum_{m=1}^{K} \frac{\rho^m}{(m-1)! \sum_{i=1}^{m} \rho^i} \Gamma(m, t)$$

OK

6i.
Consider a 3 state continuous time Markov chain as in Listing 10.6 pg 327 [SWJ-10]. Diagonalize the generator matrix Q for computing the matrix exponential by hand. Use this to obtain the distribution of the chain at time T = 0.25 and obtain results that agree with the output of Listing 10.6.

The generator matrix in question is

$$Q = \begin{pmatrix} -3 & 1 & 2 \\ 1 & -2 & 1 \\ 0 & \frac{3}{2} & -\frac{3}{2} \end{pmatrix}$$

To diagonalize this matrix we search for matrices V and D such that

$$Q = VDV^{-1}$$

Where D is a diagonal matrix. To do so we will use the spectral decomposition. The characteristic polynomial is

$$\begin{aligned}
p_\lambda =& \det(\lambda\mathbf{I} - Q) \\
=& \det\begin{pmatrix} \lambda+3 & -1 & -2 \\ -1 & \lambda+2 & -1 \\ 0 & -\frac{3}{2} & \lambda+\frac{3}{2} \end{pmatrix} \\
=& (\lambda+3)\det\begin{pmatrix} \lambda+2 & -1 \\ -\frac{3}{2} & \lambda+\frac{3}{2} \end{pmatrix} \det\begin{pmatrix} -1 & -2 \\ -\frac{3}{2} & \lambda+\frac{3}{2} \end{pmatrix} \\
=& (\lambda+3)(\lambda^2+\frac{7}{2}\lambda+\frac{3}{2}) - \lambda - \frac{9}{2} \\
=& \lambda^3 + \frac{13}{2}\lambda^2 + 11\lambda \\
=& \lambda\Big(\lambda+\frac{13}{4}+\frac{\sqrt{7}}{4}i\Big)\Big(\lambda+\frac{13}{4}-\frac{\sqrt{7}}{4}i\Big)
\end{aligned}$$

Which has the roots

$$\lambda_1 = 0 \quad \lambda_2 = -\frac{13}{4}+\frac{\sqrt{7}}{4}i \quad \lambda_3 = -\frac{13}{4}-\frac{\sqrt{7}}{4}i$$

Since this matrix has 3 unique eigenvalues it is diagonalizable. The matrix D is then

$$D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\frac{13}{4}+\frac{\sqrt{7}}{4}i & 0 \\ 0 & 0 & -\frac{13}{4}-\frac{\sqrt{7}}{4}i \end{pmatrix}$$

The eigenvector $\mathbf{v}$ corresponding to $\lambda_1 = 0$ is

$$\begin{pmatrix} 3 & -1 & -2 \\ -1 & 2 & -1 \\ 0 & -\frac{3}{2} & \frac{3}{2} \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \implies \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Which means the normalised eigenvector will be

$$\implies \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}$$

The eigenvector $\mathbf{v}$ corresponding to $\lambda_2 = -\frac{13}{4} + \frac{\sqrt{7}}{4}i$ can be found via

$$\begin{pmatrix} -\frac{1}{4} + \frac{\sqrt{7}}{4}i & -1 & -2 \\ -1 & -\frac{5}{4} + \frac{\sqrt{7}}{4}i & -1 \\ 0 & -\frac{3}{2} & -\frac{7}{4} + \frac{\sqrt{7}}{4}i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Solving the system of equations we find

$$-\frac{3}{2}v_2 + \Big(-\frac{7}{4} + \frac{\sqrt{7}}{4}i\Big)v_3 = 0 \implies v_2 = \Big(-\frac{7}{6} + \frac{\sqrt{7}}{6}i\Big)v_3$$

Substituting this into the equation in the second line of the matrix we get

$$\begin{aligned} -v_1 + \Big(-\frac{5}{4} + \frac{\sqrt{7}}{4}i\Big)v_2 - v_3 &= 0 \\ -v_1 + \Big(-\frac{5}{4} + \frac{\sqrt{7}}{4}i\Big)\Big(-\frac{7}{6} + \frac{\sqrt{7}}{6}i\Big)v_3 - v_3 &= 0 \\ -v_1 + \Big(\frac{7}{6} - \frac{\sqrt{7}}{2}i\Big)v_3 - v_3 &= 0 \\ \implies v_1 &= \Big(\frac{1}{6} - \frac{\sqrt{7}}{2}i\Big)v_3 \end{aligned}$$

Which gives the eigenvector

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = v_3 \begin{pmatrix} \frac{1}{6} - \frac{\sqrt{7}}{2}i \\ -\frac{7}{6} + \frac{\sqrt{7}}{6}i \\ 1 \end{pmatrix}$$

The norm squared of this vector is

$$\|\mathbf{v}\|^2 = \Big(\frac{1}{6} - \frac{\sqrt{7}}{2}i\Big)\Big(\frac{1}{6} + \frac{\sqrt{7}}{2}i\Big) + \Big(-\frac{7}{6} + \frac{\sqrt{7}}{6}i\Big)\Big(-\frac{7}{6} - \frac{\sqrt{7}}{6}i\Big) + 1 = \frac{1}{36} + \frac{7}{4} + \frac{49}{36} + \frac{7}{36} + 1 = 13/3$$

Therefore $\|\mathbf{v}\| = \sqrt{13/3}$, and the normalised eigenvector is

$$\hat{\mathbf{v}} = \sqrt{3/13} \begin{pmatrix} \frac{1}{6} - \frac{\sqrt{7}}{2}i \\ -\frac{7}{6} + \frac{\sqrt{7}}{6}i \\ 1 \end{pmatrix}$$

Since eigenvectors come in complex conjugate pairs, we know then that the eigenvector corresponding to $\lambda_3 = -\frac{13}{4} - \frac{\sqrt{7}}{4}i$ is

$$\hat{\mathbf{v}} = \sqrt{3/13} \begin{pmatrix} \frac{1}{6} + \frac{\sqrt{7}}{2}i \\ -\frac{7}{6} - \frac{\sqrt{7}}{6}i \\ 1 \end{pmatrix}$$

Therefore the matrix V, which has the normalised eigenvectors as columns, is

$$V = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{\sqrt{3}}{6\sqrt{13}} - \frac{\sqrt{3}\sqrt{7}}{2\sqrt{13}}i & \frac{\sqrt{3}}{6\sqrt{13}} + \frac{\sqrt{3}\sqrt{7}}{2\sqrt{13}}i \\ \frac{1}{\sqrt{3}} & -\frac{7\sqrt{3}}{6\sqrt{13}} + \frac{\sqrt{3}\sqrt{7}}{6\sqrt{13}}i & -\frac{7\sqrt{3}}{6\sqrt{13}} - \frac{\sqrt{3}\sqrt{7}}{6\sqrt{13}}i \\ \frac{1}{\sqrt{3}} & \frac{\sqrt{3}}{\sqrt{13}} & \frac{\sqrt{3}}{\sqrt{13}} \end{pmatrix}$$

The matrix exponential $P_0 e^{Q(0.25)}$ evaluate for $P_0 = (2/5, 1/2, 1/10)$ is then

$$
\begin{aligned}
P(0.25) =& P_0 V e^{D(0.25)} V^{-1} \\
=& \begin{pmatrix} \frac{2}{5} & \frac{1}{2} & \frac{1}{10} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{\sqrt{3}}{6\sqrt{13}} - \frac{\sqrt{3}\sqrt{7}}{2\sqrt{13}} i & \frac{\sqrt{3}}{6\sqrt{13}} + \frac{\sqrt{3}\sqrt{7}}{2\sqrt{13}} i \\ \frac{1}{\sqrt{3}} & -\frac{7\sqrt{3}}{6\sqrt{13}} + \frac{\sqrt{3}\sqrt{7}}{6\sqrt{13}} i & -\frac{7\sqrt{3}}{6\sqrt{13}} - \frac{\sqrt{3}\sqrt{7}}{6\sqrt{13}} i \\ \frac{1}{\sqrt{3}} & \frac{\sqrt{3}}{\sqrt{13}} & \frac{\sqrt{3}}{\sqrt{13}} \end{pmatrix} \times \\
& \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-\frac{13}{16} + \frac{\sqrt{7}}{16} i} & 0 \\ 0 & 0 & e^{-\frac{13}{16} - \frac{\sqrt{7}}{16} i} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{\sqrt{3}}{6\sqrt{13}} - \frac{\sqrt{3}\sqrt{7}}{2\sqrt{13}} i & \frac{\sqrt{3}}{6\sqrt{13}} + \frac{\sqrt{3}\sqrt{7}}{2\sqrt{13}} i \\ \frac{1}{\sqrt{3}} & -\frac{7\sqrt{3}}{6\sqrt{13}} + \frac{\sqrt{3}\sqrt{7}}{6\sqrt{13}} i & -\frac{7\sqrt{3}}{6\sqrt{13}} - \frac{\sqrt{3}\sqrt{7}}{6\sqrt{13}} i \\ \frac{1}{\sqrt{3}} & \frac{\sqrt{3}}{\sqrt{13}} & \frac{\sqrt{3}}{\sqrt{13}} \end{pmatrix}^{-1} \\
=& (0.2690, 0.4318, 0.2991)
\end{aligned}
$$

Which is equal to what was calculated in [SWJ-10]

7a.

Question 3.2, pg 101 [EM-3].

3.2 Show that in the special case of a simple stochastic epidemic starting with one infective introduced into a population of N susceptibles, the first two moments $\mathrm{E}t_1$ and $\operatorname{var} t_1$ of the duration t_1 of the process in Section 3.1 are related by

$$\operatorname{var} t_1 = \frac{1}{\beta^2(N+1)^2}\sum_{i=1}^{N}\left[\frac{1}{i} + \frac{1}{N+1-i}\right]^2 = \frac{2\beta\,\mathrm{E}t_1 + \frac{1}{3}\pi^2 + O(N^{-1})}{\beta^2(N+1)^2}\,.$$

Observe that $\operatorname{var} t_1/(\mathrm{E}t_1)^2 = [1/(N\ln N)](1+o(1))$ for large N.

TODO

7b.

Consider the SIS epidemic as in Project 2 with a > 0 using the notation of Project 2. Find an expression for stationary distribution when N = 2 (state space is {0, 1, 2}).

Assuming that $c_t(S_t, I_t, R_t) \equiv 1$ we have that this epidemic can be modelled by a continuous time Markov chain with transition rates:

$$q(i, i+1) = r_{IS}(2-i), \;\; q(i, i-1) = r_{SI}i(2-i) + a$$

Where $q(i,j) = 0$ if $(i,j) \notin \{0,1,2\}^2$. This gives rise to generator matrix:

$$Q = \begin{pmatrix} -2r_{IS} & 2r_{IS} & 0 \\ r_{SI} + a & -r_{SI} - r_{IS} - a & r_{IS} \\ 0 & a & -a \end{pmatrix}$$

The stationary distribution, π must satisfy:

$$\pi Q = \mathbf{0}$$

Where $\mathbf{0}$ is a vector of zeros. However, π must also satisfy:

$$\sum_i \pi_i = 1$$

Hence we can solve for π using:

$$\pi = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -2r_{IS} & 2r_{IS} & 1 \\ r_{SI} + a & -r_{SI} - r_{IS} - a & 1 \\ 0 & a & 1 \end{pmatrix}^{-1}$$

Where we have replaced the final column of Q with a column of all 1's and replaced the final element of $\mathbf{0}$ with a 1, then rearranged by multiplying by the matrix inverse. Performing this calculation using MATLAB's symbolic math functions we get:

$$\begin{aligned} \pi &= \begin{pmatrix} \frac{a(a+r_{SI})}{a^2+2ar_{IS}+r_{SI}a+2r_{IS}^2} & \frac{2ar_{IS}}{a^2+2ar_{IS}+r_{SI}a+2r_{IS}^2} & \frac{2r_{IS}^2}{a^2+2ar_{IS}+r_{SI}a+2r_{IS}^2} \end{pmatrix} \\ &= \frac{1}{a(a+r_{SI}) + 2ar_{IS} + 2r_{IS}^2} \begin{pmatrix} a(a+r_{SI}) & 2ar_{IS} & 2r_{IS}^2 \end{pmatrix} \end{aligned}$$

As a sanity check, it can clearly be seen that the elements of this distribution add to 1.