STAT3004/STAT7304 Sem1, 2020
Problem Set Solution Template

1a:

(a) Consider a simple branching process S_n with $S_0 = 1$ and Bernoulli-distributed offspring distribution with mean p. Determine the probability of ultimate extinction as a function of p.

1b

(b) At the start of the course, we considered examples of population models that "double every time period". The basic example was $X(t+1) = 2X(t)$ (deterministic). Then we considered simple branching processes. Refer now to Section 4.2.1 in [SP-4] and find parameters of a continuous time Branching process that "doubles every time period".

2a

(a) Consider a random walk $S_n = \sum_{i=1}^{n} X_i$ where X_i are i.i.d. random variables distributed uniformly over the set $\{-1, 0, 1\}$. Derive an expression for $\mathrm{Cov}(S_n, S_{n+k})$ with k being a positive integer.

2b Question 1.3.2, pg 18, [MC-1].

1.3.2 A gambler has £2 and needs to increase it to £10 in a hurry. He can play a game with the following rules: a fair coin is tossed; if a player bets on the right side, he wins a sum equal to his stake, and his stake is returned; otherwise he loses his stake. The gambler decides to use a bold strategy in which he stakes all his money if he has £5 or less, and otherwise stakes just enough to increase his capital, if he wins, to £10.

Let $X_0 = 2$ and let X_n be his capital after n throws. Prove that the gambler will achieve his aim with probability $1/5$.

What is the expected number of tosses until the gambler either achieves his aim or loses his capital?

3a.

Question 1.20, pg 81 [SP-1].

1.20. Three of every four trucks on the road are followed by a car, while only one of every five cars is followed by a truck. What fraction of vehicles on the road are trucks?

3b.

Question 1.33, pg 83-84 [SP-1]

1.33. In a large metropolitan area, commuters either drive alone (A), carpool (C), or take public transportation (T). A study showed that transportation changes according to the following matrix:

	A	**C**	**T**
A	.8	.15	.05
C	.05	.9	.05
S	.05	.1	.85

In the long run what fraction of commuters will use the three types of transportation?

3c.
Question 1.48, pg 88 [SP-1].

1.48. Six children (Dick, Helen, Joni, Mark, Sam, and Tony) play catch. If Dick has the ball, he is equally likely to throw it to Helen, Mark, Sam, and Tony. If Helen has the ball, she is equally likely to throw it to Dick, Joni, Sam, and Tony. If Sam has the ball, he is equally likely to throw it to Dick, Helen, Mark, and Tony. If either Joni or Tony gets the ball, they keep throwing it to each other. If Mark gets the ball, he runs away with it. (a) Find the transition probability and classify the states of the chain. (b) Suppose Dick has the ball at the beginning of the game. What is the probability Mark will end up with it?

3d.
Question 1.57, pg 90 [SP-1].

1.57. 3. Two barbers and two chairs. Consider the following chain

	0	**1**	**2**	**3**	**4**
0	0	1	0	0	0
1	0.6	0	0.4	0	0
2	0	0.75	0	0.25	0
3	0	0	0.75	0	0.25
4	0	0	0	0.75	0.25

(a) Find the stationary distribution. (b) Compute $P_x(V_0 < V_4)$ for $x = 1, 2, 3$.
(c) Let $\tau = \min\{V_0, V_4\}$. Find $E_x\tau$ for $x = 1, 2, 3$.

3e.

Question 1.65, pg 92 [SP-1].

1.65. *Ehrenfest chain.* Consider the Ehrenfest chain, Example 1.2, with transition probability $p(i, i+1) = (N-i)/N$, and $p(i, i-1) = i/N$ for $0 \leq i \leq N$. Let $\mu_n = E_x X_n$. (a) Show that $\mu_{n+1} = 1 + (1 - 2/N)\mu_n$. (b) Use this and induction to conclude that

$$\mu_n = \frac{N}{2} + \left(1 - \frac{2}{N}\right)^n (x - N/2)$$

From this we see that the mean μ_n converges exponentially rapidly to the equilibrium value of $N/2$ with the error at time n being $(1 - 2/N)^n (x - N/2)$.

3f.

Question 1.73, pg 94 [SP-1].

1.73. Consider the Markov chain with state space $\{1, 2, \ldots\}$ and transition probability

$$\begin{aligned} p(m, m+1) &= m/(2m+2) && \text{for } m \geq 1 \\ p(m, m-1) &= 1/2 && \text{for } m \geq 2 \\ p(m, m) &= 1/(2m+2) && \text{for } m \geq 2 \end{aligned}$$

and $p(1, 1) = 1 - p(1, 2) = 3/4$. Show that there is no stationary distribution.

3g.

Question 1.5.4, pg 29 [MC-1].

1.5.4 A random sequence of non-negative integers $(F_n)_{n\geq 0}$ is obtained by setting $F_0 = 0$ and $F_1 = 1$ and, once $F_0, \dots, F_n$ are known, taking F_{n+1} to be either the sum or the difference of F_{n-1} and F_n, each with probability 1/2. Is $(F_n)_{n\geq 0}$ a Markov chain?

By considering the Markov chain $X_n = (F_{n-1}, F_n)$, find the probability that $(F_n)_{n\geq 0}$ reaches 3 before first returning to 0.

Draw enough of the flow diagram for $(X_n)_{n\geq 0}$ to establish a general pattern. Hence, using the strong Markov property, show that the hitting probability for $(1,1)$, starting from $(1,2)$, is $(3-\sqrt{5})/2$.

Deduce that $(X_n)_{n\geq 0}$ is transient. Show that, moreover, with probability 1, $F_n \to \infty$ as $n \to \infty$.

3h.
Question 1.8.4, pg 46 [MC-1].

1.8.4 Each morning a student takes one of the three books he owns from his shelf. The probability that he chooses book i is α_i, where $0 < \alpha_i < 1$ for $i = 1, 2, 3$, and choices on successive days are independent. In the evening he replaces the book at the left-hand end of the shelf. If p_n denotes the probability that on day n the student finds the books in the order 1,2,3, from left to right, show that, irrespective of the initial arrangement of the books, p_n converges as $n \to \infty$, and determine the limit.

3i.

Consider Theorem 1.8.3 on pg 41 [MC-1]. Rewrite the theorem and the proof specializing to two state Markov chains on the state space $\{1, 2\}$.

Theorem 1.8.3 (Convergence to equilibrium). *Let P be irreducible and aperiodic, and suppose that P has an invariant distribution π. Let λ be any distribution. Suppose that $(X_n)_{n\geq 0}$ is Markov(λ, P). Then*

$$\mathbb{P}(X_n = j) \to \pi_j \quad \text{as } n \to \infty \text{ for all } j.$$

In particular,

$$p_{ij}^{(n)} \to \pi_j \quad \text{as } n \to \infty \text{ for all } i, j.$$

To understand this proof one should see what goes wrong when P is not aperiodic. Consider the two-state chain of Example 1.8.1 which has $(1/2, 1/2)$ as its unique invariant distribution. We start $(X_n)_{n\geq 0}$ from 0 and $(Y_n)_{n\geq 0}$ with equal probability from 0 or 1. However, if $Y_0 = 1$, then, because of periodicity, $(X_n)_{n\geq 0}$ and $(Y_n)_{n\geq 0}$ will never meet, and the proof fails. We move on now to the cases that were excluded in the last theorem, where $(X_n)_{n\geq 0}$ is periodic or transient or null recurrent. The remainder of this section might be omitted on a first reading.

4a.

Derive the chain binomial representation in Equation (1.4.2) of pg 12 [EM-1].

$$
\begin{aligned}
L &= P\{X_1, Y_1 \mid X_0, Y_0\}\, P\{X_2, Y_2 \mid X_1, Y_1\}\, P\{X_2, 0 \mid X_2, Y_2\} \\
&= \binom{X_0}{X_1}(1-q^{Y_0})^{Y_1} q^{Y_0 X_1} \binom{X_1}{X_2}(1-q^{Y_1})^{Y_2} q^{Y_1 X_2} \binom{X_2}{X_2}(1-q^{Y_2})^0 q^{Y_2 X_2} \\
&= \binom{X_0}{X_1}\binom{X_1}{X_2}(1-q^{Y_0})^{Y_1}(1-q^{Y_1})^{Y_2} q^{Y_0 X_1 + Y_1 X_2 + Y_2 X_2},
\end{aligned}
\tag{1.4.2}
$$

4b.

Derive Equations (4.1.4) pg 107 [EM-4]. This was part of Project 1 as well.

$$\begin{aligned} \mathbf{E}[X_t \mid X_0 = x_0] &= \alpha^t x_0, \\ \mathbf{E}[Y_t \mid X_0 = x_0] &= \alpha^{t-1}(1-\alpha)x_0. \end{aligned} \tag{4.1.4}$$

5a.

Question 2.23, pg 119 [SP-2].

2.23. *When did the chicken cross the road?* Suppose that traffic on a road follows a Poisson process with rate λ cars per minute. A chicken needs a gap of length at least c minutes in the traffic to cross the road. To compute the time the chicken will have to wait to cross the road, let $t_1, t_2, t_3, \ldots$ be the interarrival times for the cars and let $J = \min\{j : t_j > c\}$. If $T_n = t_1 + \cdots + t_n$, then the chicken will start to cross the road at time T_{J-1} and complete his journey at time $T_{J-1} + c$. Use the previous exercise to show $E(T_{J-1} + c) = (e^{\lambda c} - 1)/\lambda$.

5b.
Question 2.30, pg 120 [SP-2].

2.30. Suppose that the number of calls per hour to an answering service follows a Poisson process with rate 4. Suppose that 3/4's of the calls are made by men, 1/4 by women, and the sex of the caller is independent of the time of the call. (a) What is the probability that in one hour exactly two men and three women will call the answering service? (b) What is the probability 3 men will make phone calls before three women do?

5c.

Question 2.43, pg 122 [SP-2].

2.43. A policewoman on the evening shift writes a Poisson mean 6 number of tickets per hour. 2/3's of these are for speeding and cost \$100. 1/3's of these are for DWI and cost \$400. (a) Find the mean and standard deviation for the total revenue from the tickets she writes in an hour. (b) What is the probability that between 2 a.m. and 3 a.m. she writes five tickets for speeding and one for DWI. (c) Let A be the event that she writes no tickets between 1 a.m. and 1:30, and N be the number of tickets she writes between 1 a.m. and 2 a.m. Which is larger $P(A)$ or $P(A|N = 5)$? Don't just answer yes or no, compute both probabilities.

5d.

Question 2.47, pg 123 [SP-2].

2.47. Let S and T be exponentially distributed with rates λ and μ. Let $U = \min\{S, T\}$ and $V = \max\{S, T\}$. Find (a) EU. (b) $E(V - U)$, (c) EV. (d) Use the identity $V = S + T - U$ to get a different looking formula for EV and verify the two are equal.

5e.

Consider the algorithm for generating a Poisson random variable with mean λ using Equation (3.15) pg 100 [SWJ-3]. Explain (prove) why this algorithm works.

The Poisson process possesses many elegant analytic properties, and these sometimes come as an aid when considering Poisson distributed random variables. One such (seemingly magical) property is to consider the random variable $N \geq 0$ such that,

$$\prod_{i=1}^{N} U_i \geq e^{-\lambda} > \prod_{i=1}^{N+1} U_i, \tag{3.15}$$

where $U_1, U_2, \ldots$ is a sequence of i.i.d. uniform$(0,1)$ random variables and $\prod_{i=1}^{0} U_i \equiv 1$. It turns out that seeking such a random variable N produces an efficient recipe for generating a Poisson random variable. That is, the N defined by (3.15) is Poisson distributed with mean λ. Notice that the recipe dictated by (3.15) is to continue multiplying uniform random variables to a "running product" until the product goes below the desired level $e^{-\lambda}$.

5f.

Consider Theorem 2.4.6 pg 79 [MC-2]. Rewrite the theorem and the proof, specializing to the case of $n = 2$. Plot/sketch the joint distribution of $J1$ and $J2$ in this case, with $X10 = 2$.

6a.
Question 4.7, pg 193 [SP-4].

4.7. Two people are working in a small office selling shares in a mutual fund. Each is either on the phone or not. Suppose that calls come in to the two brokers at rate $\lambda_1 = \lambda_2 = 1$ per hour, while the calls are serviced at rate $\mu_1 = \mu_2 = 3$. (a) Formulate a Markov chain model for this system with state space $\{0, 1, 2, 12\}$ where the state indicates who is on the phone. (b) Find the stationary distribution. (c) Suppose they upgrade their telephone system so that a call to one line that is busy is forwarded to the other phone and lost if that phone is busy. Find the new stationary probabilities. (d) Compare the rate at which calls are lost in the two systems.

6b.
Question 4.14, pg 194 [SP-4].

4.14. A small company maintains a fleet of four cars to be driven by its workers on business trips. Requests to use cars are a Poisson process with rate 1.5 per day. A car is used for an exponentially distributed time with mean 2 days. Forgetting about weekends, we arrive at the following Markov chain for the number of cars in service:

	0	**1**	**2**	**3**	**4**
0	–1.5	1.5	0	0	0
1	0.5	–2.0	1.5	0	0
2	0	1.0	–2.5	1.5	0
3	0	0	1.5	–3	1.5
4	0	0	0	2	–2

(a) Find the stationary distribution. (b) At what rate do unfulfilled requests come in? How would this change if there were only three cars? (c) Let $g(i) = E_i T_4$. Write and solve equations to find the $g(i)$. (d) Use the stationary distribution to compute $E_3 T_4$.

6c.
Question 4.20, pg 195 [SP-4].

4.20. Consider an $M/M/4$ queue with no waiting room, for example, four prostitutes hanging out on a street corner. Customers arrive at times of a Poisson process with rate 4. Each service takes an exponentially distributed amount of time with mean 1/2. If no server is free, then the customer goes away never to come back. (a) Find the stationary distribution. (b) At what rate do customers enter the system? (c) Use $W = L/\lambda_a$ to calculate the average time a customer spends in the system.

6d.

Consider the M/M/∞ queue with arrival rate λ and service rate μ. Determine the variance of the stationary distribution of the number of customers in the queue.

6e Question 3.6.3, pg 123 [MC-3].

3.6.3 Customers arrive at a single-server queue in a Poisson stream of rate λ. Each customer has a service requirement distributed as the sum of two independent exponential random variables of parameter μ. Service requirements are independent of one another and of the arrival process. Write down the generator matrix Q of a continuous-time Markov chain which models this, explaining what the states of the chain represent. Calculate the essentially unique invariant measure for Q, and deduce that the chain is positive recurrent if and only if $\lambda/\mu < 1/2$.

6f

Consider the two state continuous time Markov chain with birth rate λ and death rate μ. Denote the state space $\{0, 1\}$. Write the forward equations and the backward equations and use one of these to find expressions for $\mathbb{P}(X_t = 1 | X_t = 0)$.

6g.

Consider the M/M/1/K queue where finite K is the total system capacity (state space is {0, 1, 2, . . . , K}). Assume λ 6= μ. Find an expression for the mean number of customers in the system in steady state.

6h.

For the M/M/1/K Try to reproduce the computations for P(W > x) as in pg 335 [SWJ-10] to obtain some expression for the CCDF of the waiting time.

$$\begin{aligned}
\mathbb{P}(\mathrm{W} > x) &= \sum_{k=1}^{\infty} \mathbb{P}(\mathrm{W} > x \mid X = k)\mathbb{P}(X = k) \\
&= \sum_{k=1}^{\infty} \int_x^{\infty} f_k(u)\, du\ (1-\rho)\rho^k \\
&= \sum_{k=1}^{\infty} \int_x^{\infty} \frac{\mu^k}{(k-1)!} u^{k-1} e^{-\mu u}\, du\ (1-\rho)\rho^k \\
&= (1-\rho)\lambda \int_x^{\infty} e^{-\mu u} \sum_{k=0}^{\infty} \frac{(\lambda u)^k}{k!}\, du \\
&= (1-\rho)\lambda \int_x^{\infty} e^{-(\mu-\lambda)u}\, du \\
&= (1-\rho)\frac{\lambda}{\mu-\lambda} e^{-(\mu-\lambda)x} \\
&= \rho e^{-(\mu-\lambda)x}.
\end{aligned}$$

6i.
Consider a 3 state continuous time Markov chain as in Listing 10.6 pg 327 [SWJ-10]. Diagonalize the generator matrix Q for computing the matrix exponential by hand. Use this to obtain the distribution of the chain at time T = 0.25 and obtain results that agree with the output of Listing 10.6.

7a.
Question 3.2, pg 101 [EM-3].

3.2 Show that in the special case of a simple stochastic epidemic starting with one infective introduced into a population of N susceptibles, the first two moments $\mathrm{E}t_1$ and $\operatorname{var} t_1$ of the duration t_1 of the process in Section 3.1 are related by

$$\operatorname{var} t_1 = \frac{1}{\beta^2(N+1)^2}\sum_{i=1}^{N}\left[\frac{1}{i}+\frac{1}{N+1-i}\right]^2 = \frac{2\beta\,\mathrm{E}t_1 + \frac{1}{3}\pi^2 + O(N^{-1})}{\beta^2(N+1)^2}\,.$$

Observe that $\operatorname{var} t_1/(\mathrm{E}t_1)^2 = [1/(N \ln N)](1+o(1))$ for large N.

7b.

Consider the SIS epidemic as in Project 2 with $a > 0$ using the notation of Project 2. Find an expression for stationary distribution when $N = 2$ (state space is $\{0, 1, 2\}$).