1. Let \( q(x, y) \) denote the rate of an \( x \to y \) transition, for \( y \neq x \). For states \( n \geq 2 \), the system behaves as a simple queue with arrival rate \( \nu \) and service rate \( \mu := \mu_A + \mu_B \). Thus, for \( n \geq 2 \), \( q(n, n + 1) = \nu \) and, for \( n \geq 3 \), \( q(n, n - 1) = \mu \). For the other states we have

\[
q(1A, 0A) = q(2, 1B) = \mu_A, \quad q(1B, 0B) = q(2, 1A) = \mu_B,
\]

\[
q(0A, 1A) = q(1A, 2) = q(0B, 1B) = q(1B, 2) = \nu.
\]

All other transition rates are zero. The transition diagram is a tree.

Since the transition diagram is a tree, the detail-balance equations have a unique solution. We should therefore solve

\[
\pi(0A)\nu = \pi(1A)\mu_A, \quad \pi(0B)\nu = \pi(1B)\mu_B,
\]

\[
\pi(1A)\nu = \pi(2)\mu_B, \quad \pi(1B)\nu = \pi(2)\mu_A,
\]

and

\[
\pi(n)\nu = \pi(n + 1)\mu, \quad \text{for } n = 2, 3, \ldots
\]

Hence \( \pi(n) = \pi(2)\rho^{n-2} \), for \( n \geq 2 \), where \( \rho = \nu/\mu < 1 \), and

\[
\pi(1A) = \pi(2)\mu_B/\nu, \quad \pi(0A) = \pi(1A)\mu_A/\nu = \pi(2)\mu_A\mu_B/\nu^2,
\]

\[
\pi(1B) = \pi(2)\mu_A/\nu, \quad \pi(0B) = \pi(1B)\mu_B/\nu = \pi(2)\mu_A\mu_B/\nu^2.
\]

An equilibrium distribution exists because \( \rho < 1 \) and, in order to normalize \( \pi \), we choose \( \pi(2) = 1/C \), where

\[
C = \frac{2\mu_A\mu_B}{\nu^2} + \frac{\mu_A + \mu_B}{\nu} + \sum_{n=2}^{\infty} \rho^{n-2} = \frac{2\mu_A\mu_B}{\nu^2} + \frac{\mu_A + \mu_B}{\nu} + \frac{1}{1 - \rho}
\]

\[
= \frac{2\mu_A\mu_B}{\nu^2} + \frac{\mu_A + \mu_B}{\nu} + \frac{\mu_A + \mu_B}{\mu_B - \nu}.
\]

Finally, let \( \pi(n) \) be the probability that there are \( n \) customers present in the system. We have already evaluated \( \pi(n) \) for \( n \geq 2 \). The remaining values are

\[
\pi(0) = \pi(0A) + \pi(0B) = \frac{2\mu_A\mu_B}{\nu^2}\pi(2)
\]

and

\[
\pi(1) = \pi(1A) + \pi(1B) = \frac{\mu_A + \mu_B}{\nu}\pi(2).
\]

2. The state space is \( S_M = \{ m \in \mathbb{Z}_+^M : \sum_{i=1}^{M} im_i = M \} \). The detailed balance equations are

\[
\pi(m)q(m, m - e_i - e_i + e_{i+1}) = \pi(m - e_i - e_i + e_{i+1})q(m - e_i - e_i + e_{i+1}, m) \quad (i \geq 2)
\]

\[
\pi(m)q(m, m - 2e_1 + e_2) = \pi(m - 2e_1 + e_2)q(m - 2e_1 + e_2, m),
\]

and so we must verify that

\[
\pi(m)\alpha m_i m_i = \pi(m - e_i - e_i + e_{i+1})(i + 1) \beta(m_{i+1} + 1) \quad (i \geq 2)
\]

\[
\pi(m)\alpha m_1 (m_1 - 1) = \pi(m - 2e_1 + e_2)2 \beta(m_2 + 1).
\]

Solutions to Assignment 6

(These solutions outline the main steps: you will need to provide full details)
The right-hand side of (1) is 
\[
\pi(m) \frac{m_1}{\rho} \frac{m_i}{(\rho/i!)} \frac{(\rho/(i+1)!)}{m_{i+1}+1} (i+1)\beta(m_{i+1}+1) = \pi(m) \frac{\beta}{\rho} m_im_i = \pi(m) \alpha m_im_i,
\]
which is the left-hand side of (1). Similarly, the right-hand side of (2) is 
\[
\pi(m) \frac{m_1(m_1-1)}{\rho^2} \frac{(\rho/2)}{m_2+1} 2\beta(m_2+1) = \pi(m) \frac{\beta}{\rho} m_1(m_1-1) = \pi(m) \alpha m_1(m_1-1),
\]
which is the left-hand side of (2).

With isolates being allowed to enter and leave the party, the state space is now the countable subset \( S \) of \( \mathbb{Z}_+^n \) whose members \( m = (m_1, m_2, \ldots) \) have only finitely many positive entries. The extra non-zero transition rates are 
\[
q(m, m+e_1) = \lambda \\
q(m, m-e_1) = \mu m_1.
\]
So, for the chain to be reversible, we require \( \pi(m)\lambda = \pi(m+e_1)\mu(m_1+1) \), that is, 
\[
\pi(m) = \pi(m+e_1) \frac{m_1+1}{\rho}.
\]
This is clearly satisfied by the equilibrium distribution of the closed system but normalized now over \( S \). Therefore, 
\[
\pi(m) = \text{constant} \times \prod_{i=1}^{\infty} \frac{(\rho/i)!}{m_i!} = \prod_{i=1}^{\infty} e^{-\rho/i!} \frac{(\rho/i)!}{m_i!},
\]
and we deduce that the equilibrium numbers \( m_1, m_2, \ldots \) are independent with \( m_i \) have a Poisson distribution with mean \( \rho/i! \).
Postgraduate problem: Since \( S \) is finite the equilibrium distribution is the stationary distribution, and thus the reverse transition function is well defined: \( p^*_{kj}(t) = \pi_j p_{jk}(t)/\pi_k \). But, for \( \tau > 0 \),
\[
p_k(t + \tau) = \sum_{j \in S} p_j(t) p_{jk}(t),
\]
and so
\[
\frac{p_k(t + \tau)}{\pi_k} = \sum_{j \in S} \frac{p^*_{kj}(\tau) p_j(t)}{\pi_j},
\]
Finally, using the second part of the hint, we see that, for \( \tau > 0 \),
\[
H(t + \tau) = \sum_{k \in S} \pi_k h \left( \frac{p_k(t + \tau)}{\pi_k} \right) = \sum_{k \in S} \pi_k h \left( \sum_{j \in S} \frac{p^*_{kj}(t) p_j(t)}{\pi_j} \right)
\]
\[
> \sum_{j \in S} \sum_{k \in S} \pi_k p^*_{kj}(t) h \left( \frac{p_j(t)}{\pi_j} \right) = \sum_{j \in S} \sum_{k \in S} \pi_j p_{jk}(t) h \left( \frac{p_j(t)}{\pi_j} \right) = H(t),
\]
so that \( H \) is strictly increasing. It has limit \( h(1) \) because and \( h \) is continuous and \( p_j(t) \to \pi_j \).
In Question 2 of Assignment 5, we found that \( P(t) = A + e^{-\alpha t} B \), where \( \alpha = 5 \),
\[
A = \frac{1}{5} \begin{pmatrix} 2 & 1 & 2 \\ 2 & 1 & 2 \\ 2 & 1 & 2 \end{pmatrix}, \quad \text{and} \quad B = \frac{1}{5} \begin{pmatrix} 3 & -1 & -2 \\ -2 & 4 & -2 \\ -2 & -1 & 3 \end{pmatrix},
\]
and that the limiting distribution was \( \pi = \left( \frac{2}{3}, \frac{1}{3}, \frac{2}{3} \right) \). Thus, with initial distribution \( p(0) = \left( \frac{1}{4}, \frac{1}{4}, \frac{1}{2} \right) \), the state probability vector is \( p(t) = \frac{1}{20} (8 - 3e^{-5t} 4 + e^{-5t} 8 + 2e^{-5t}) \). The following graph plots \( H(t) \) versus \( t \) in the case when \( h(x) = -x \log x \):
\[
H(t) = -\sum_{k=1}^{3} p_k(t) \log \left( \frac{p_k(t)}{\pi_k} \right).
\]
We see that \( H(t) \) increases from \( H(0) = -\frac{1}{4} \left( \log \left( \frac{5}{8} \right) + 3 \log \left( \frac{5}{4} \right) \right) \approx -0.049857 \) to near its limit, 0 (= \( h(1) \)), at \( t = 1 \).
Remark: When \( h(x) = -x \log x \), \( H \) is called the entropy of \( p(t) \) with respect to \( \pi \); entropy increases as a Markov chain moves towards its equilibrium.