Continuous and Discrete random process

- Discrete stochastic processes.
- Continuous stochastic process taking values in $\Re$.

Many real data falls into the continuous category: Meteorological data, molecular motion, traffic data ...

Brownian process

- In 1827 the English botanist Robert Brown $\rightarrow$ pollen grains suspended in water moved around following a zigzag path.
- More remarkable was the fact that pollen grains that had been stored for a century moved in the same way.

Brownian motion in action!

Brownian process

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Brownian motion in action!

Robert Brown called this movement 'Brownian motion', but he couldn't work out what was causing it.

- In 1904 Einstein $\rightarrow$ Brownian motion was due to molecules of water hitting the tiny pollen grains
- Einstein $\rightarrow$ it was possible to work out how many molecules were hitting a single pollen grain and how fast the water molecules were moving.

Diffusion

A particle is diffusing about a space $\Re^n$ if it experience erratic an disordered motion through the space.
Some examples are:

- Radioactive particles diffusing through the atmosphere.
- A rumour diffusing through a population.

For the time being we will focus on one dimensional diffusions $\rightarrow$ The position of the observed particle at any time is a point in $\Re$

Brownian motion

Takes place in continuous time and continuous space.
The first attempt to model it $\rightarrow$ By approximating it by a discrete process $\rightarrow$ Random walk:

- At any time the position of a observed particle is contained to move about $\{(a\delta, b\delta, c\delta) : a, b, c = 0, +-1, +-1, ...\}$ of a three-dimensional cubit lattice.
- The distance between neighbouring points is $\delta \rightarrow$ fixed positive number (very small)
- Suppose that the particle performs a symmetric random walk on the lattice
- The position s_n after n jumps

$$P(S_{n+1} = S_n + \delta\epsilon) = 1/6$$

if $\epsilon = (+-1, 0, 0), (0, +-1, 0), (0, 0, +-1)$

Brownian motion

- Looking at the x coordinate of the particle $S_n = (S_n^1, S_n^2, S_n^3)$

$$S_n^1 - S_0^1 = \sum_{i=1}^{n} X_i$$

- X_i is an independent identically distributed sequence $P(X_i = k\delta) \rightarrow$ *whiteboard*

We are interested in the displacement of $S_n^1 - S_0^1$ when n is large. In that case the CLT $\rightarrow$ the distribution of the displacement is approximately $N(0, \frac{1}{3}n\delta^2)$

Brownian motion

- Suppose that the jumps of the random walk → $\tau, 2\tau, 3\tau, \ldots$ with $\tau > 0$ time in between jumps (very small) ⇒ A very large number of jumps occur in any time interval → the particle after some time $t > 0$ has elapsed.
- By the time the particle elapsed it has experienced $n = [t/\tau]$ jumps so its x coordinate is such that $S^1(t) - S^1(0) \sim N(0, \frac{1}{3}t\delta^2/\tau)$.
- By letting τ and δ go to zero the discrete random walk maybe approach some limit whose properties have something in common with the observe features of Brownian motion.
- If $\tau, \delta \to 0$ then $S^1(t) - S^1(0)$ approaches $N(0, \sigma^2 t)$ where $\sigma^2 = \frac{1}{3}\delta^2/\tau$.
- Same arguments will follow for y and z

Brownian motion

Mathematica Simulation

Wiener process

This is a diffusion process and can be used to model the random displacement of Brownian motion in a rigours manner $\rightarrow$ sometimes also called Brownian Motion..

Wiener process

It is a stochastic process $W = \{W_t : t \geq 0\}$ characterise by the following properties:

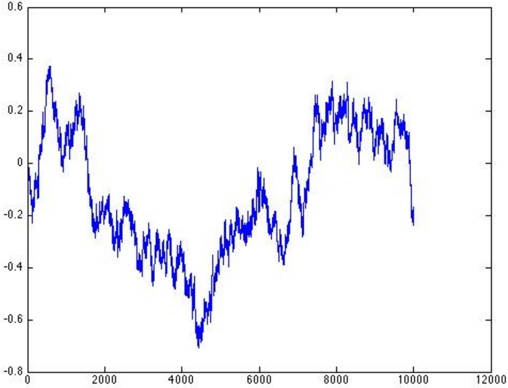
- W has independent increments:
  For any $t_1 < t_2 \leq t_3 < t_4$ $W_{t_4} - W_{t_3}$ and $W_{t_2} - W_{t_1}$ are independent random variables.
- $W(s+t) - W(s) \sim N(0, \sigma^2 t)$ for all $s, t \geq 0$ and $\sigma^2 > 0$
- The sample paths of W are continuous.

Wiener process: Independent increments

The wiener process W has stationary independent increments, that is:

- the distribution of $W(t) - W(s)$ depends on $t - s$ alone
- the variables $W(t_j) - W(s_j)$, $1 \leq j \leq n$ are independent whenever the intervals $(s_j, t_j]$ are disjoint

Wiener process



Wiener process

Two kind of statement can be made about diffusion processes in general:

- Sample path properties
- Distributional properties

- Wiener process is a Markov process
- Central role in probability
- Forms the basis of many other stochastic processes
- It can be viewed as a continuous version of a random walk

Wiener process

Autocovariance function of W

$Cov(W_s, W_t) = \sigma^2 s + 0$ if $0 \leq s \leq t$ which is equivalent to

$$Cov(W_s, W_t) = \sigma^2 min\{s, t\}$$

for all $s, t \geq 0$

W is continuous in mean square

$$E([W(s+t) - W(s)]^2) \to 0$$

as $t \to 0$.

Wiener process simulation in Matlab

Algorithm

1. Select $t_0 < t_1 < \ldots < t_n$ times for the process simulations.
2. Generate $Z_1, \ldots Z_n \sim N(0,1)$ iid random variables and compute

$$W_{t_k} = \sum_{i=1}^{k} \sqrt{t_k - t_{k-1}} Z_i$$

for $k = 1, \ldots, n$.

Please note that this algorithm returns a discrete realisation of the process.
Matlab simulation.

Wiener process simulation

The previous algorithm returns only a discrete sample path. If you would want to obtain a continuous path to approximate the exact path of the Wiener process, you could use linear interpolation on the points $W_{t_1}, \ldots, W_{t_n}$. That is, in each interval $[t_{k-1}, t_k]$, $k = 1, \ldots, n$ approximate the continuous process $\{W_x, s \in [t_{k-1}, t_k]\}$ via:

$$W_s = \frac{W_{t_k}(s - t_{k-1}) + W_{t_{k-1}}(t_k - s)}{(t_k - t_{k-1})}$$

Reference: Handbook of Monte Carlo Methdos.

Distribution of the Wiener process

- Supose that $W(s) = x$ for $s \geq 0$ and $x \in \Re$.
- Conditional on that $\mathcal{L}(W(t)) \sim N(x, t-s)$ for $t \geq s$

Probability distribution and density functions

- $F(t, y|s, x) = P(W(t) \leq y|W(s) = x)$
- $f(t, y|s, x) = \frac{1}{\sqrt{2\pi(t-s)}} exp\big(-\frac{(y-x)^2}{2(t-s)}\big)$ for $-\infty < y < \infty$

Forward and backwards equations

The function below is a function of four variables however it can been seen as

$$f(t, y|s, x) = \frac{1}{\sqrt{2\pi(t-s)}} exp\Big(-\frac{(y-x)^2}{2(t-s)}\Big)$$

for $-\infty < y < \infty$

a solution of the forward and backwards equations:

$$\frac{\partial f}{\partial t} = \frac{1}{2}\frac{\partial^2 f}{\partial y^2}$$

$$\frac{\partial f}{\partial s} = \frac{-1}{2}\frac{\partial^2 f}{\partial x^2}$$

Certain boundary conditions need to be specified.

Forward and backwards equations

The above derivatives have coefficients which are independent of $x, y, s, t \rightarrow$ Wiener process is homogeneous in space and time.

- The increment $W(t) - W(s)$ is independent of $W(s)$ for all $t \geq s$.
- The increments are stationary in time.

- Wiener process $\rightarrow$ is a Markov process $\rightarrow$ Forward and Backward equations.
- Similar forward and backward equations exist $\rightarrow$ the coefficients will not be constant.

Non-homegeneous diffusion processes

Let $D = \{D(t) : t \geq 0\}$ a diffusion process with continuous sample paths (a.s).
We need conditions to specify the mean and variance of the increments $D(t+h) - D(t)$ over small time intervals $(t, t+h)$.

Suppose that there exist functions $a(t,x)$ (drift) and $b(t,x)$ (instantaneous variance)such that:

$$P(|D(t+h) - D(t)| > \epsilon | D(t) = x) = o(h) \text{ for all } \epsilon > 0$$
$$E(D(t+h) - D(t) | D(t) = x) = a(t,x)h + o(h)$$
$$E([D(t+h) - D(t)]^2 | D(t) = x) = b(t,x)h + o(h)$$

Non-homegeneous diffusion processes

Again subject to some technical conditions, if $s \leq t$ then the conditional density of $D(t)$ given $D(s) = x$

$$f(t, y|s, x) = \frac{\partial f}{\partial y} P(D(t) \leq y|D(s) = x)$$

and satisfies the following partial differential equations

$$\begin{aligned} \frac{\partial f}{\partial t} &= -\frac{\partial f}{\partial y}[a(t, y)] + \frac{1}{2}\frac{\partial^2}{\partial y^2}[b(t, y)f] \\ \frac{\partial f}{\partial s} &= -a(s, x)\frac{\partial f}{\partial x} - \frac{1}{2}b(s, x)\frac{\partial^2 f}{\partial x^2} \end{aligned}$$

$f \rightarrow$ specified as soon as the instantaneous mean a and variance b are known. We don't need any further information about the distribution of the increment. This is very convenient in many applications where a and b are specified in a natural manner by the physical description of the process.