# Wiener process: Brownian motion



## Wiener process

It is a stochastic process $W = \{W_t : t \geq 0\}$ with the following properties:

- $W$ has **independent increments**:
  For all times $t_1 \leq t_2 \ldots \leq t_n$ the random variables $W_{t_n} - W_{t_{n-1}}, W_{t_{n-1}} - W_{t_{n-2}}, \ldots, W_{t_2} - W_{t_1}$ are independent random variables.
- It has **stationary increments**: The distribution of the increment $W(t+h) - W(t)$ does not depende on t.
- $W(s+t) - W(s) \sim N(0, \sigma^2 t)$ for all $s, t \geq 0$ and $\sigma^2 > 0$.
- The process $W = \{W_t : t \geq 0\}$ has almost surely **continuos sample paths**.

# Example:

Suppose that $W$ is a Brownian motion or Wiener process and $U$ is an independent random variable which is uniformly distributed on $[0,1]$. Then the process

$$\widetilde{W} = \begin{cases} W(t), & \text{if } t \neq U \\ 0, & \text{if } t = U \end{cases}$$

- Same marginal distributions as a Wiener process.
- Discountinuous if $W(U) \neq 0$ with probability one.

Hence this process is not a Brownian motion. The continuity of sample paths is essential for Wiener process $\rightarrow$ cannot jump over any valule $x$ but must pass through it!

# Wiener process: Brownian motion



- The process $W$ is called standard Wiener process if $\sigma^2 = 1$ and if $W(0) = 0$.
- Note that if $W$ is non-standard $\rightarrow$ $W_1(t) = (W(s) - W(0))/\sigma$ is standard.
- We also have seen that $W$ $\rightarrow$ **Markov property/ Weak Markov property**:
  If we know the process $W(t) : t \geq 0$ on the interval $[0, s]$, for the prediction of the future $\{W(t) : t \geq s\}$, this is as useful as knowing the endpoint $X(s)$.
- We also have seen that $W$ $\rightarrow$ **Strong Markov property**:
  The same as above holds even when $s$ is a random variable if $s$ is a stopping time.
- Reflexion principle

# Reflexion principle and other properties

- First passage times $\rightarrow$ stopping times.
  First time that the Brownian process hits a certain value
  - Density function of the stopping time $T(x)$
- We studied properties about the maximum of the Wiener process:
  - The random variable $M(t) = max\{W(s) : 0 \leq s \leq t\} \rightarrow$ same law as $|W(t)|$.
  - We studied the probability that the standard Wiener returns to its origin in a given interval

# Properties when the reflexion principle does not hold



The study of first passage times $\rightarrow$ lack of symmetry properties for the diffusion process

- We learnt how to definite a martingale based on a diffusion process:

$$U(t) = e^{-2mD(t)} \rightarrow \ martingale$$

- Used that results to find the distribution of the first passage times of $D$

# Barriers

- Diffusion particles $\rightarrow$ have a restricted movement due to the space where the process happends.
- Pollen particles where contained in a glass of water for instance.

What happend when a particle hits a barrier?

- Same as with random walks we have two situations:
  - Absorbing
  - Reflecting

# Example: Wiener process



- Let $W$ be the standard Wiener process.
- Let $w \in \Re^{+}$ positive constant.
- We consider the shifted process $w + W(t)$ which starts at $w$.

**Wiener process $W^a$ absorbed at 0**

$$W^a(t) = \begin{cases} w + W(t), & \text{if } t \leq T \\ 0, & \text{if } t \geq T \end{cases}$$

with $T = inf\{t : w + W(t) = 0\}$ being the hitting time of the position 0.

$W^r(t) = W^r(t) = |w + W(t)|$ is the Wiener process reflected at 0.

# Example: Wiener process

- $W^a$ and $W^r$ satisfy the forward and backward equations, if they are away from the barrier.
- In other words, $W^a$ and $W^r$ are diffusion processes.

**Transition density for $W^a$ and $W^r$?**

Solving the diffusion equations subject to some suitable boundary conditions.

# Example: Transition densities for the Wiener process



Diffusion equations for the Wiener process:

Let $f(t, y)$ denote the density function of the random variable $W(t)$ and consider $W^a$ and $W^r$ as before.

- The density function of $W^a(t)$. is

$$f^a(t, y) = f(t, y - w) - f(t, y + w), \;\; y > 0$$

- The density function of $W^r(t)$ is

$$f^r(t, y) = f(t, y - w) + f(t, y + w), \;\; y > 0.$$

where the funtioon $f(t, y)$ is the $N(0, t)$ density function.

# Example: Wiener process with drift



Suppose that we are looking into the Wiener process with drift so that
$a(t,x) = m$ and $b(t,x) = 1$ for all $t$ and $x$.

- Suppose that there is an absorbing barrier at 0.
- Suppose $D(0) = d > 0$

**Aim : find a solution g(t,y) to the foward equation**

$$\frac{\partial g}{\partial t} = -m\frac{\partial g}{\partial y} + \frac{1}{2}\frac{\partial^2 g}{\partial y^2}$$

for $y > 0$ subject to

$$\begin{aligned} g(t,0) &= 0,\ t \geq 0 \\ g(0,y) &= \delta_d(y)\ , y \geq 0 \end{aligned}$$

with $\delta_d$*to* Dirac $\delta$ centered at $d$.

# Example: Wiener process with drift

We saw that the Wiener process with drift is the solution of the forward and backward equations and we saw that in general

$$g(t,x|x) = \frac{1}{\sqrt{2\pi t}} exp\Big( - \frac{(y-x-mt)^2}{2t}\Big)$$

Now what we need is to find a linear combination of such functions $g(\cdot,\cdot|x)$ which satisfy the boundary conditions.

**Solution:**

$$f^a(t,y) = g(t,y|d) - e^{-2md} g(t,y|-d); \;\; y > 0.$$

Assuming uniqueness, that is the density function of $D^a(t)$.

# Example: Wiener process with drift



Now let's see how is the density function of the time $T$ until the absorption of the particle.

- At time $t$ either the process has been absorbed or its position has density

$$f^a(t,y) = g(t,y|d) - e^{-2md} g(t,y|-d); \;\; y > 0.$$

$$P(T \leq t) = 1 - \int_0^\infty f^a(t,y)dy = 1 - \Phi(\frac{mt+d}{\sqrt{t}}) + e^{-2md}\Phi(\frac{mt-d}{\sqrt{t}})$$

Taking derivatives:

$$f_T(t) = \frac{d}{\sqrt{2\pi t^3}} exp(-\frac{(d+mt)^2}{2t}), \;\; t > 0$$

& $P(absorption\ take\ place) = P(T < \infty) = \begin{cases} 1, & \text{if } m \leq 0 \\ e^{-2md}, & \text{if } m > 0 \end{cases}$

# Browinian Bridge



We are interested in properties of the Wiener process conditioned on special events.

**Question**

What is the probability that $W$ has no zeros in the time interval $(0, v]$ given that it has none in the smaller interval $(0, u]$?

Here, we are considering the Wiener process $W = \{W(t) : t \geq 0\}$ with $W(0) = w$ and $\sigma^2 = 1$.

# Browinian Bridge



We are interested in properties of the Wiener process conditioned on special events.

**Question**

What is the probability that $W$ has no zeros in the time interval $(0, v]$ given that it has none in the smaller interval $(0, u]$?

If $w \neq 0$ then the answer is

$$P(no\ zeros\ in\ (0, v]|W(0) = w)/P(no\ zeros\ in\ (0, u]|W(0) = w)$$

we can compute each of those probabilities by using the distribution of the maxima.

# Browinian Bridge



## If $w = 0$ then both numerator and denominator $\to 0$

$$lim_{w\to 0}\frac{P(no\ zeros\ in\ (0,v]|W(0)=w)}{P(no\ zeros\ in\ (0,u]|W(0)=w)} =$$

$$lim_{w\to 0}\frac{g_w(v)}{g_w(u)}$$

where $g_w(x) \to$ is the probability that a Wiener process starting at $W$ fails to reach 0 at time $x$. It can be shown by using the symmetry priciple and the theorem for the density of $M(t)$ that

$$g_w(x) = \sqrt{\frac{2}{\pi x}}\int_0^{|w|} exp(-frac12m^2/x)dm.$$

Then $g_w(v)/g_w(u) \to \sqrt{u/v}$ $as\ w \to 0$

# Excursion



An "excursion" of $W$ is a trip taken by $W$ away from 0

## Definition

If $W(u) = W(v) = 0$ and $W(t) \neq 0$ for $u < t < v$ then the trajectory of $W$ during the interval $[u, v]$ is called an excursion of the process.

Excursions are positive if $W > 0$ throughout $(u, v)$ and negative otherwise.

# Martingales and Excursions

**Excursions martingale**

Let $Y(t) = \sqrt{Z(t)} sign\{W(t)\}$ and $\mathcal{F}_t = sigma(\{Y(u) : 0 \leq u \leq t\})$. Then $(Y, \mathcal{F})$ is a martingale.

The probability that the standard Wiener process $W$ has a positive excursion of total duration at least $a$ before it has a negative excursion of total duration at least $b$ is $\sqrt{b}/(\sqrt{a} + \sqrt{b})$.

# Brownian Bridge



Let $B = \{B(t) : 0 \leq t \leq 1\}$ be a process with continuous sample paths and the same fdds as $\{W(t) : 0 \leq t \leq 1\}$ conditoned on $W(0) = W(1) = 0$. The process $B$ is a diffusion process with drift $a$ and instantaneous variance $b$ given by $a(t,x) = -\frac{x}{1-t}$ and $b(t,x) = 1$, $x \in \Re$, $0 \leq t \leq 1$.

The Brownian Bridge has the same instantaneous variance as $W$ but its drift increasing in magnitude as $t \to 1$ and it has the effect of guiding the process to its finishing point $B(1) = 0$

# Stochastic differential equations and Diffusion Processes



A stochastic differential equation for a stochastic process $\{X_t, t \geq 0\}$ is an expression of the form

$$dX_t = a(X_t, t)dt + b(X_t, t)dWt$$

where $\{W_t, t \geq 0\}$ is a Wiener process and $a(x, t)$ (drift) and $b(x, t)$ (diffusion coefficient) are deterministic functions.

- $\{X_t, t \geq 0\}$ is a Markov process with continuous sample paths $\rightarrow$ it is an Itô diffusion.

Stochastic differential equations share similar principles as ordinary differential equations by relating an unknown function to its derivatives but with the difference that part of the unknown function includes randomness.

# Stochastic differential equations and the Chain rule

We are going to see how to derive a differential equation as the one before.

- Consider the process $X_t = f(W_t)$ to be a function of the standard Wiener process.
- The standard chain rule $\rightarrow$ $dX_t = f^{'}(W_t)dW_t$ $\rightarrow$ incorrect in this contest.

If $f$ is sufficiently smooth by Taylor's theorem

$$X_{t+\delta t} - X_t = f^{'}(W_t)(\delta W_t) + \frac{1}{2}f^{''}(W_t)(\delta W_t)^2) + \ldots$$

where $\delta W_t = W_{t+\delta t} - W_t$

- In the usual chain rule $\rightarrow$ it is used $W_{t+\delta t} - W_t = o(\delta t)$.
- However in the case here $(\delta W_t)^2$ has mean $\delta t$ so we can not applied the statement above.

# Stochastic differential equations and the Chain rule

## Solution

We approximate $(\delta W_t)^2$ by $\delta t \Rightarrow$ the subsequent terms in the Taylor expansion are insignificant in the limit as $\delta t \to 0$

$$dX_t = f^{'}(W_t)dW_t + \frac{1}{2}f^{''}(W_t)dt$$

being that an special case of the Ito'formula and

$$X_t - X_0 = \int_0^t f'(W_s)dW_s + \int_0^t \tfrac{1}{2}f^{''}(W_s)ds$$

# Stochastic differential equations and Diffusion Processes



$$dX_t = a(X_t, t)dt + b(X_t, t)dWt$$

Expresses the infinitesimal change in $dX_t$ at time $t$ as the sum of infinitesimal displacement $a(X_t, t)dt$ and some noise $b(X_t, t)dW_t$.

## Mathematically

The stochastic process $\{X_t, t \geq 0\}$ satisfies the integral equation

$$X_t = X_0 + \int_0^t a(X_s, s)dx + \int_0^t b(X_s, s)dWs.$$

The last integral is the so called Ito integral.

# Stochastic calculus and Diffusion Processes



We have seen the diffusion process $D = \{D_t : t \geq 0\}$ as a Markov process with continuous sample paths having "instantaneous mean" $\mu(t, x)$ and "instantaneous variance" $\sigma(t, x)$.

- The most standard and fundamental diffusion process is the Wiener process

  $$W = \{W_t : t \geq 0\}$$

  with instantaneous mean 0 and variance 1.

$$dD_t = \mu(t, D_t)dt + \sigma(t, D_t)dW_t$$

which is equivalent to

$$D_t = D_0 = \int_0^t \mu(s, D_s)dx + \int_0^t \sigma(s, D_s)dWs$$

# Example: Geometric Wiener process



Suppose that $X_t$ is the price from some stock or commodity at time $t$.

How can we represent the change $dX_t$ over a small time interval $(t, t + dt)$?

If we assume that changes in the price are proportional to the price and otherwise they appear to be random in sign and magnitude as the movements of a molecule. we can model this by

$$dX_t = bX_t dW_t$$

or by

$$X_t - X_0 = \int_0^t bX_s dW_s$$

for some constant b. This is called the geometric Wiener process.

# Interpretation of the stochastic integral



Let's see how we can interprete

$$\int_0^t W_s dW_s$$

- Consider $t = n\delta$ with $\delta$ being small and positve.
- We partition the interval $(0, t]$ into intervals $(j\delta, (j+1)\delta]$ with $0 \leq j < n$.
- If we take $\theta_j \in [j\delta, (j+1)\delta]$, we can consider

$$I_n = \sum_{j=0}^{n-1} W_{\theta_j}\left(W_{(j+1)\delta} - W_{j\delta}\right)$$

- If we think about the Riemann integral $\rightarrow$ $W_{j\delta}$, $W_{\theta_j}$ and $W_{(j+1)\delta}$ should be close to one antoher for $I_n$ to have a limit as $n \rightarrow \infty$ independent of the choice of $\theta_j$

# Interpretation of the stochastic integral

- However, in our case, the Wiener process $W$ has sample paths with unbounded variation.
- It is easy to see

$$2I_n = W_t^2 - W_0^2 - Z_n$$

where $Z_n = \sum_{j=0}^{n-1}(W_{(j+1\delta)} - W_{j\delta})^2$

- Implying $E(Z_n - t)^2 \to 0$ as $n \to \infty$ ($Z_n \to t$ in mean square).
- So that $I_n \to \frac{1}{2}(W_t^2 - t)$ in mean square as $n \to \infty$

$$\int_0^t W_s dW = \frac{1}{2}(W_t^2 - t)$$

That is an example of an Ito Integral